String structures, 3-forms, and tmf classes

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Consider $P \xrightarrow{\pi} M$, where P is a principal $Spin(k)$-bundle over a closed manifold M (compact without boundary). We then define a (topological) string structure on P to be a lift of the classifying map from $BSpin(k)$ to $BString(k)$. Here, $BString(k)$ is the homotopy fiber of the characteristic class $\frac{p_1}{2}$, as seen in the fibration sequence

$$BString(k) \to BSpin(k) \xrightarrow{\frac{p_1}{2}} K(\mathbb{Z}, 4).$$

While there are various descriptions of string structures, any construction will produce such a lift, and homotopy classes of lifts to $BString(k)$ have a convenient classification.

Definition. *A string class $\mathcal{S}$ is a cohomology class $\mathcal{S} \in H^3(P;\mathbb{Z})$ that restricts fiberwise to the stable generator of $H^3(Spin(k);\mathbb{Z})$.*

Proposition.

- *$\{$String structures$\}/($homotopy$) \cong \{$String classes$\}$*
- *A string structure/class exists if and only if $\frac{p_1}{2}(P) = 0 \in H^4(M;\mathbb{Z})$.*
- *The set of string classes is a torsor for $H^3(M;\mathbb{Z})$ under the natural additive action of π^*.*

We now wish to describe the harmonic representative of a string class. A Riemannian metric on P determines the Hodge Laplacian Δ acting on differential forms. Hodge's Theorem implies that $\operatorname{Ker}\Delta^i$, the harmonic i-forms, is canonically isomorphic to $H^i(P;\mathbb{R})$.

Start with the data $(P \xrightarrow{\pi} M, g, A)$, where g is a Riemannian metric on M, and A is a connection on P. The connection A provides an orthogonal splitting of TP. Then, the choice of a bi-invariant metric g_{Spin} on $Spin(k)$ defines the 1-paremeter family of Riemannian metrics on P

$$g_\delta := \pi^* g \oplus \delta^2 g_{Spin}$$

for $\delta > 0$. Shrinking the fibers, or taking the limit as $\delta \to 0$, is known as the adiabatic limit. While the metric becomes singular at $\delta = 0$, work of Mazzeo–Melrose, Dai, and Forman [MM, Dai, For] show that the harmonic forms extend smoothly to $\delta = 0$. We denote this limit as

$$\mathcal{H}^i(P) := \lim_{\delta \to 0} \operatorname{Ker}\Delta^i_{g_\delta} \subset \Omega^i(P)$$

and note that $\mathcal{H}^i(P) \cong H^i(P;\mathbb{R})$.

Theorem. *Consider $(P \xrightarrow{\pi} M, g, A)$ with $\frac{p_1}{2}(P) = 0$. In the adiabatic limit, the harmonic representative of a string class $\mathcal{S}$ is of the form $CS_3(A) - \pi^* H_{\mathcal{S},g,A}$, where $CS_3(A)$ is the Chern–Simons 3-form, and $H_{\mathcal{S},g,A} \in \Omega^3(M)$; i.e.*

$$\begin{array}{ccc} H^3(P;\mathbb{Z}) \to H^3(P;\mathbb{R}) & \xrightarrow{\cong} & \mathcal{H}^3(P) \\ \mathcal{S} \longmapsto & & CS_3(A) - \pi^* H_{\mathcal{S},g,A}, \end{array}$$

(If one does not take the adiabatic limit, the difference between $CS_3(A)$ and the harmonic representative of $\mathcal{S}$ is *not* in general in $\pi^*\Omega^3(M)$.) This form $H_{\mathcal{S},g,A}$ satisfies two useful properties. First,

$$d^* H_{\mathcal{S},g,A} = 0 \in \Omega^2(M).$$

Secondly, the connection A determines a differential cohomology class $\frac{\check{p_1}}{2}(A)$ [CS], and $H_{\mathcal{S},g,A} = \frac{\check{p_1}}{2}(A)$ as differential classes. This is encoded in the following standard exact sequence:

$$\begin{gathered}\Omega^3_{\mathbb{Z}}(M) \to \Omega^3(M) \to \check{H}^4(M) \to H^4(M;\mathbb{Z}) \to 0 \\ H_{\mathcal{S},g,A} \mapsto \frac{\check{p_1}}{2}(A) \mapsto \frac{p_1}{2}(P) = 0\end{gathered}$$

In the language of differential characters, $\frac{\check{p_1}}{2}(A)$ is a homomorphism from 3-cycles to $\mathbb{R}/\mathbb{Z}$, and the form $H_{\mathcal{S},g,A}$ gives a specified lift of the homomorphism to $\mathbb{R}$. There is also the following equivariance: if one changes the string class by adding $\pi^*\psi \in \pi^*H^3(M;\mathbb{Z})$, then

$$H_{\mathcal{S}+\pi^*\psi,g,A} = H_{\mathcal{S},g,A} + H_{\psi,g}$$

where $H_{\psi,g}$ is the harmonic representative of ψ. This changes the lift of the character from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$ in the expected way. The above story can be duplicated with $Spin(k)$ replaced by any compact, simply-connected, semi-simple Lie group G, and $\frac{p_1}{2}$ replaced by a level $\lambda \in H^4(BG;\mathbb{Z})$.

Our motivation for dealing with string structures stems from

$$MString \xrightarrow{\sigma} tmf,$$

the *String*-orientation of the cohomology theory tmf or topological modular forms [Hop]. A spin manifold M^n with string class $\mathcal{S} \in H^3(Spin(TM);\mathbb{Z})$ naturally produces an element in string-bordism and a class $\sigma(M,\mathcal{S}) \in tmf^{-n}(pt)$ refining the Witten genus. The Witten genus is, heuristically, the S^1-equivariant index of $\not{D}_{LM}$, the Dirac operator on the free loop space LM. The string structure actually arises when constructing the mathematically rigorous spinor bundle on LM. It is hoped that the natural home for families index theorems on loop spaces will live in tmf, just as ordinary families index theorems live in K and KO-theory. The analogy between the Witten genus and the $\widehat{A}$-genus led Stolz to the following conjecture.

Conjecture (Stolz [Sto])**.** *Let M^n be a spin manifold with $\frac{p_1}{2}(M) = 0 \in H^4(M;\mathbb{Z})$. If M admits a metric of positive Ricci curvature, then the Witten genus of M is zero.*

One could also ask if something analogous to Hitchin's theorem might hold. Namely, if a string manifold M^n admits a positive Ricci curvature metric, then is $\sigma(M^n,\mathcal{S}) = 0 \in tmf^{-n}$? While there are no known counterexamples to Stolz' conjecture, the answer to the preceding question is most certainly no.

For example, consider $S^3 \cong SU(2)$. The Witten genus is 0 (it does not have dimension $4k$), yet the various framings produce different string structures which yield all elements in

$$MString^{-3}(pt) \cong tmf^{-3}(pt) \cong \pi_3^s \cong \mathbb{Z}/24.$$

Furthermore, the round metric on S^3 has positive Ricci curvature, and even positive sectional curvature. So, any attempt to generalize Hitchin's theorem must take into account both the geometry and the string structure. This leads to the following hypothesis, where $H_{\mathcal{S},g}$ is the 3-form constructed above with $P = Spin(TM)$ and A the Levi–Civita connection.

Hypothesis. *Let M^n be a spin manifold with $\frac{p_1}{2}(M) = 0 \in H^4(M;\mathbb{Z})$. If M admits a string class and metric $(\mathcal{S}, g)$ such that g has positive Ricci curvature and $H_{\mathcal{S},g} = 0$, then $\sigma(M,\mathcal{S}) = 0 \in tmf^{-n}(pt)$.*

The condition that $H_{\mathcal{S},g} = 0$ is quite strong as it implies that the differential class $\frac{\check{p}_1}{2}(g) = 0$. If we consider the situation of S^3, there is a useful 1-parameter family of left-invariant metrics, known as the Berger metrics, obtained by rescaling the fibers of the Hopf fibration. The above hypothesis holds true in this family of metrics, yet it would not if either condition were weakened. In particular, when g is the round metric and $\mathcal{S}$ is induced from D^4, the form $H = 0$ and the σ-invariant is 0. However, there is a metric for which the Ricci curvature is nonnegative but not positive; this metric and the right-invariant framing produce $H = 0$ and a generator of $tmf^{-3}(pt)$. There are also infinitely many other string classes and metrics which produce $H = 0$ but not positive Ricci curvature.

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