HOMEWORK 8 - DUE 11/16/2015

MATH 511

(1) In this problem, we will show how to construct the rational numbers from multiplication on the integers. You may assume all standard properties of $\mathbb{Z}$ and multiplication, but you are not allowed to use division.

Define the relation $\sim$ on $\mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$ by

$$(a, b) \sim (c, d) \iff a \cdot d = b \cdot c.$$

(a) Prove that $\sim$ is an equivalence relation.

We now let $\mathbb{Q}$ denote the quotient set $(\mathbb{Z} \times (\mathbb{Z}\backslash\{0\}))/\sim$. A fraction p/q is given by the equivalence class $[(p, q)]$.

(b) Prove that multiplication in $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$ is well-defined via the formula

$$[(a, b)] \cdot [(c, d)] = [(ac, bd)].$$

In other words, you must show the following: Suppose that $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$, then $(a_1c_1, b_1d_1) \sim (a_2c_2, b_2d_2)$.

(c) Prove that $[(1, 1)] \in \mathbb{Q}$ is a multiplicative unit; i.e. $[(1, 1)]x = x$ for all $x \in \mathbb{Q}$.

(2) Show that there are the same "number" of integers as even integers.

(3) In Hilbert's infinite hotel, there are countably many rooms and no vacancies. Show that one can accommodate a countably infinite number of new guests, despite not having any vacant rooms. Can you find multiple ways of accommodating the new guests?

(4) Let $a, b \in \mathbb{R}$ with $a < b$. Construct a bijection $[0, 1] \xrightarrow{\approx} [a, b]$. Construct a bijection $(0, 1) \xrightarrow{\approx} (a, b)$. Construct a bijection $[0, 1) \xrightarrow{\approx} (0, 1]$.

(5) Prove that $\mathbb{R} \approx (0, 1)$. (*Hint:* Start with any function $(a, b) \to \mathbb{R}$ that is a bijection.)

(6) Show that $[0, 1] \approx (0, 1]$. (*Hint:* Let $A \subset [0, 1]$ be a countable subset, and map every element in A to the next element.)

(7) Use problems 4-6 to show that all intervals in $\mathbb{R}$ are equivalent as sets.

Date: November 9, 2015.