HOMEWORK 10 - DUE 11/29/17

MATH 511

(1) (Optional) Consider the relation R on $\mathbb{R}$ given by
$$R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$
Determine whether R is reflexive, symmetric, antisymmetric, and/or transitive.

(2) (Optional) Give an example of a relation on $\mathbb{R}$ that is symmetric but not reflexive.

(3) (Already done in class) Consider the relation on $\mathbb{Z}$ defined: $a \equiv b \mod n$ if there exists $t \in \mathbb{Z}$ such that $b - a = nt$.
  (a) Show that $(\bullet \equiv \bullet \mod n)$ is an equivalence relation on $\mathbb{Z}$.
  (b) Begin to list the elements in the graph of $\equiv$ whose first number is 0.

(4) Consider the relation $\equiv_5$ on $\mathbb{Z}$ defined by:
$$a \equiv_5 b \iff \exists t \in \mathbb{Z} \text{ such that } b - a = 5t.$$
From the previous problem, it follows that this is an equivalence relation. We will denote the quotient set $\mathbb{Z}/5\mathbb{Z}$.
  (a) What is the subset of $\mathbb{Z}$ denoted by $[0]$? By $[2]$?
  (b) Find a system of equivalence class representatives for $\mathbb{Z}/5\mathbb{Z}$; i.e. say that $\mathbb{Z}/5\mathbb{Z} = \{[\bullet], \dots, [\bullet]\}$.
  (c) Prove that $[a] + [b] := [a + b]$ gives a well-defined addition on $\mathbb{Z}/5\mathbb{Z}$. Specifically, you need to check: If $[a_1] = [a_2]$ and $[b_1] = [b_2]$, then $[a_1 + b_1] = [a_2 + b_2]$.
  (d) Calculate the following: $[0] + [3]$, $[1] + [3]$, and $[3] + [4]$.

(5) Calculate the following quantities by hand.
  (a) $1492 \cdot 3478 \mod 2$
  (b) $6 + 25 \mod 7$
  (c) $2^{25} \mod 7$
  (d) $5^{23} \mod 7$
  (e) A biology experiment starts at 2 p.m. and lasts 80 hours. At what time of the day will the experiment end?

(6) Let X, Y be sets, and let $f : X \to Y$ be a function. Define the relation $\sim_f$ on X by
$$x_1 \sim_f x_2 \iff f(x_1) = f(x_2).$$
Prove that $\sim_f$ is an equivalence relation.

Date: November 15, 2017.

(7) Consider the function sign : $\mathbb{R} \to \mathbb{R}$ defined by the following.

$$\operatorname{sign}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

By problem (2), the relation $\sim_{\text{sign}}$ is an equivalence relation.

(a) When will $x \sim_{\text{sign}} y$?

(b) What is the quotient set $\mathbb{R}/\sim_{\text{sign}}$?

(8) (Save for next week) In this problem, we will show how to construct the rational numbers from multiplication on the integers. You may assume all standard properties of $\mathbb{Z}$ and multiplication, but you are not allowed to use division.

Define the relation $\sim$ on $\mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$ by

$$(a, b) \sim (c, d) \iff a \cdot d = b \cdot c.$$

(a) Prove that $\sim$ is an equivalence relation.

We now let $\mathbb{Q}$ denote the quotient set $(\mathbb{Z} \times (\mathbb{Z}\backslash\{0\}))/\sim$. A fraction p/q is given by the equivalence class $[(p, q)]$.

(b) Prove that multiplication in $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$ is well-defined via the formula

$$[(a, b)] \cdot [(c, d)] = [(ac, bd)].$$

In other words, you must show the following: Suppose that $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$, then $(a_1c_1, b_1d_1) \sim (a_2c_2, b_2d_2)$.

(c) Prove that $[(1, 1)] \in \mathbb{Q}$ is a multiplicative unit; i.e. $[(1, 1)]x = x$ for all $x \in \mathbb{Q}$.

(9) Suppose there are 150 people in the International Club, and that 55 speak German, 42 speak French, 74 speak Spanish. Suppose 30 speak French and German, 35 speak German and Spanish, 29 speak French and Spanish, and 21 speak all 3 languages. Determine how many people speak:

(a) French or German

(b) Spanish or French

(c) Spanish but not French

(d) Neither Spanish or French

(e) French or German or Spanish

(f) (Spanish and French) or German