HOMEWORK 11 - DUE 12/6/17

MATH 511

(1) In this problem, we will show how to construct the positive rational numbers from multiplication on the positive integers. You may assume all standard properties of $\mathbb{Z}$ and multiplication, but you are not allowed to use division.

Define the relation $\sim$ on $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ by

$$(a, b) \sim (c, d) \iff a \cdot d = b \cdot c.$$

(a) Prove that $\sim$ is an equivalence relation.

We now let $\mathbb{Q}_{>0}$ denote the quotient set $(\mathbb{Z}_{>0} \times \mathbb{Z}_{>0})/_{\sim}$. A fraction p/q is given by the equivalence class $[(p, q)]$.

(b) Prove that multiplication in $(\mathbb{Z}_{>0} \times \mathbb{Z}_{>0})/_{\sim}$ is well-defined via the formula

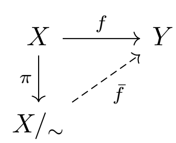
$$[(a, b)] \cdot [(c, d)] = [(ac, bd)].$$

In other words, you must show the following: Suppose that $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$, then $(a_1c_1, b_1d_1) \sim (a_2c_2, b_2d_2)$.

(c) Prove that $[(1, 1)] \in \mathbb{Q}$ is a multiplicative unit; i.e. $[(1, 1)]x = x$ for all $x \in \mathbb{Q}$.

(2) (I talked about this in class two weeks ago, but I am asking you to supply a complete proof now.) Suppose $\sim$ is an equivalence relation on X, and $f\colon X \to Y$ is a function. Let $\pi\colon X \to X/_{\sim}$ be the natural quotient map.

(a) Show there exists a function $\bar{f}\colon X/_{\sim} \to Y$ satisfying $\bar{f} \circ \pi = f$ if and only if $f(x_1) = f(x_2)$ whenever $x_1 \sim x_2$.



(b) Show that if such an $\bar{f}$ exists, it is unique.

Date: November 29, 2017.