

# Math 512 Homework 4: Due Feb 27, 2013

February 20, 2013

1. Let  $<$  be a relation on a set  $S$  that satisfies the following two properties

- Transitivity: If  $a < b$  and  $b < c$ , then  $a < c$  for all  $a, b, c \in S$ .
- Trichotomy: For all  $a, b \in S$ , exactly one of the following holds:  $a < b$ ,  $b < a$ , or  $a = b$ .

(Such a relation is called a strict total order.)

Show, using induction, that if  $S$  is *finite*, then there is a “maximal” element  $M \in S$  that satisfies the following: for all  $x \in S$ , if  $x \neq M$ , then  $x < M$ .

2. Write an algorithm to compute the function  $f : \mathbb{N}^2 \rightarrow \mathbb{N}$ ,  $f(x, y) = x^y$ . You may use the arithmetic operations of addition, subtraction, multiplication, and division.
3. Write a Turing machine that adds two numbers.
4. (Bonus) How could you write a Turing machine that would multiply two numbers?
5. (Bonus) Is there a way to do composition of Turing machines? In other words, given a Turing machines  $T_f$  and  $T_g$  that calculate  $f : \mathbb{N} \rightarrow \mathbb{N}$  and  $g : \mathbb{N} \rightarrow \mathbb{N}$ , can you create a Turing machine that calculates  $f \circ g$ ? What if generalize to  $g : \mathbb{N}^j \rightarrow \mathbb{N}^k$  and  $f : \mathbb{N}^k \rightarrow \mathbb{N}^l$ ?