Obtaining Initial Simplex Form Solution?
To run the simplex method, we start from a Linear Program (LP) in the following standard simplex form.

\[
\begin{align*}
\text{Max} & \quad z \\
\text{s.t.} & \quad (-z) + a_{01}x_1 + \cdots + a_{0n}x_n = b_0 \\
& \quad a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\
& \quad \vdots \\
& \quad a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \\
& \quad x_i \geq 0
\end{align*}
\]
Max \[ z \]

s.t. \[-z + a_{01}x_1 + \cdots + a_{0n}x_n = b_0\]
\[ a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \]
\[ \vdots \]
\[ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \]
\[ x_i \geq 0 \]

To be in standard simplex form:
1. All decision variables (except for \(-z\)) are non-negative.
2. All other constraints are equalities.
3. The RHS (except for the “cost row” or “z-row”) is non-negative.
4. For each row \(i\), there is a column equal to \(e_i\) (a 1 in row \(i\), and 0 in all other rows).
There are multiple conventions as to what constitutes “Standard Form.” They are all different, but more or less equivalent in terms of requirements.

Given an LP in the form: Max $z$, subject to inequalities all of the form $\sum a_i x_i \leq b$, one only needs to introduce slack variables to obtain starting standard form for Simplex Method.

Today, we will learn techniques for more complicated LPs.
Example 1

Is the following LP in standard simplex form?
Maximize $z$, subject to $x_1, x_2, x_3, s_1, s_2 \geq 0$ and the equalities:

<table>
<thead>
<tr>
<th></th>
<th>$-z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$= $</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>9</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>15</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Non-negative decision variables? ✓
- Equalities for constrains? ✓
- Non-negative RHS entries? ✓
- Columns $e_i$? ✓ $-z, s_1, s_2$
Example 2

Is the following LP in standard simplex form?
Maximize $z$, subject to $x_1, x_2, x_3, s_1, s_2 \geq 0$ and the equalities:

\[
\begin{array}{ccccccc}
-\text{z} & x_1 & x_2 & x_3 & s_1 & s_2 & = \text{RHS} \\
1 & 1 & 9 & 1 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & -1 & 0 & 9 \\
0 & 3 & 2 & 2 & 0 & 1 & 15
\end{array}
\]

- Non-negative decision variables? ✓
- Equalities for constrains? ✓
- Non-negative RHS entries? ✓
- Columns $e_i$? $\times$ No column $[0 \ 1 \ 0]^T$
Potential complications

1. Minimizing instead of maximizing.
2. Decision variables allowed to take negative values.
3. Inequalities of form $\sum a_i x_i \geq b$. 
Maximize/Minimize

Maximizing $f(x_1, \ldots, x_n) \iff$ Minimizing $-f(x_1, \ldots, x_n)$

Notes:

- To change between max/min, just multiply all coefficients by $-1$.
- In practice, some implementations of simplex method assume you are maximizing, and some assume you are minimizing. Max vs Min is a minor detail.
- When doing simplex method by hand, you may simply keep cost row the same as when maximizing, but perform pivots on columns with a negative entry in the cost row.
Bounded below: Suppose we have a variable $x \geq -20$.

Then: Substitute for new variable $x = \hat{x} - 20$, or $\hat{x} = x + 20$, with $\hat{x} \geq 0$.

Note: For constraint $x \geq 20$, we may introduce surplus variable, or we may use substitution $x = \hat{x} + 20$, with $\hat{x} \geq 0$.

Unbounded: Suppose we have unbounded variable $w$.

Then: Use substitution $w = w^+ - w^-$, with $w^+, w^- \geq 0$.

Note: When more than one decision variable is unrestricted, a single variable $x^-$ can be used for all of them, with the interpretation that it is the most negative of all the decision variables.
Put LP in Simplex Form to start Simplex Method

Min $x + y$

s.t.

$x + y \leq 20$
$x + y \geq -20$
$x - y \leq 20$
$x - y \geq -20$

$x, y$ unrestricted

**Question:** Will the LP have a unique solution?
Inequalities:

\[-20 \leq x + y \leq 20\]
\[-20 \leq x - y \leq 20\]

Substitutions:

\[x = x^+ - x^- \text{ and } y = y^+ - y^-\]
\[x^+, x^-, y^+, y^- \geq 0\]

New inequalities:

\[-20 \leq x^+ - x^- + y^+ - y^- \leq 20\]
\[-20 \leq x^+ - x^- - y^+ + y^- \leq 20\]
Min  \( x^+ - x^- + y^+ - y^- \)

s.t.  \( x^+ - x^- + y^+ - y^- \leq 20 \)

\( x^+ - x^- + y^+ - y^- \geq -20 \)

\( x^+ - x^- - y^+ + y^- \leq 20 \)

\( x^+ - x^- - y^+ + y^- \geq -20 \)

\( x^+, x^-, y^+, y^- \geq 0 \)
Max \quad z = -x^+ + x^- - y^- + y^- \\
\text{s.t.} \quad x^+ - x^- + y^+ - y^- \leq 20 \\
\quad x^+ - x^- + y^+ - y^- \geq -20 \\
\quad x^+ - x^- - y^+ + y^- \leq 20 \\
\quad x^+ - x^- - y^+ + y^- \geq -20 \\
\quad x^+, x^-, y^+, y^- \geq 0
Max \[ z = -x^+ + x^- - y^- + y^- \]
\[
\text{s.t.} \quad x^+ - x^- + y^+ - y^- + s_1 = 20 \\
 x^+ - x^- + y^+ - y^- - s_2 = -20 \\
 x^+ - x^- - y^+ + y^- + s_3 = 20 \\
 x^+ - x^- - y^+ + y^- - s_4 = -20 \\
 x^+, x^-, y^+, y^- \geq 0 \]
Maximize $z$ subject to non-negativity constraints and:

\[
\begin{array}{cccccccccc}
(-z) & x^+ & x^- & y^+ & y^- & s_1 & s_2 & s_3 & s_4 & RHS \\
1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 20 \\
0 & 1 & -1 & 1 & -1 & 0 & -1 & 0 & 0 & -20 \\
0 & 1 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 20 \\
0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & -1 & -20 \\
\end{array}
\]
Alternate Options:

- Since two variables are unbounded, we could substitute $x = x^+ - t$ and $y = y^+ - t$, where $x^+, y^+, t \geq 0$. Here, $t = \max\{-x, -y, 0\}$.

- Inspecting the inequalities, we observe that $x, y \geq -20$. Hence, we could use substitutions $x = \hat{x} - 20, \ y = \hat{y} - 20, \ \hat{x}, \hat{y} \geq 0$. 
Inequalities

Given: \( \sum_j a_{ij}x_j \geq b_i \), with \( b_i > 0 \)

Introduce: surplus variable \( s \geq 0 \) to form

\[
\sum_j a_{ij}x_j - s_i = b_i.
\]

Problem: Neither \( \sum_j a_{ij}x_j - s_i = b_i \) nor \( \sum_j (-a_{ij}x_j + s_i = -b_i \) give something in simplex form.

Solution: Introduce *artificial variable(s)* \( \alpha_i \geq 0 \),
\[
\sum_j a_{ij}x_j - s_i + \alpha_i = b_i.
\]

Then, solve the related LP with objective function \( \sum \alpha_i \), and same constraints. This produces an initial Basic Feasible Solution and immediately translates to the LP in simplex form.
Max \[2x_1 + x_2\]
\begin{align*}
\text{s.t.} & \quad x_1 + x_2 \leq 3 \\
& \quad -x_1 + x_2 \geq 1 \\
& \quad x_1, x_2 \geq 0
\end{align*}
Phase I auxiliary LP:

Min $x_5$

s.t. $x_1 + x_2 + x_3 = 3$

$-x_1 + x_2 - x_4 + x_5 = 1$

$x_1, \ldots, x_5 \geq 0$

\[
\begin{array}{cccccc|c}
-w & x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 3 \\
0 & -1 & 1 & 0 & -1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc|c}
-w & x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\
1 & -1 & 1 & 0 & -1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 3 \\
0 & -1 & 1 & 0 & -1 & 1 & 1 \\
\end{array}
\]
Solution to Phase I auxiliary LP:

<table>
<thead>
<tr>
<th></th>
<th>$-w$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Solution of $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 0$ gives $x_5 = 0$.
These values $(x_1, \ldots, x_4)$ satisfy the original LPs constraints, and they form our initial Basic Feasible Solution!

Phase II, Solving initial LP:

<table>
<thead>
<tr>
<th></th>
<th>$-Z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

ERO’s to isolate $x_2$, $x_3$, then in simplex form.
Suppose that when performing the simplex method, you obtain column with positive number in objective row, and non-positive numbers in rest of column. Then, the feasible region is unbounded, and a solution does not exist.

**Example:**

\[
\begin{array}{cccccc}
-z & x_1 & x_2 & x_3 & x_4 & \text{RHS} \\
1 & 2 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 1 & 1 & 2 \\
0 & -1 & 1 & 0 & -1 & 1 \\
\end{array}
\]

For *basic solution*, we let \(x_1 = 0\). But, we can let \(x_1\) be any positive number, and we obtain a *better* feasible solution.
Cycling

When several iterations of the simplex method do not improve the current objective value, this is called **stalling**.

When, after several iterations, the simplex method returns a previous tableau, this is called **cycling**.

In general, stalling and cycling can occur. Some implementations of the simplex method include special provisions to prevent cycling; other implementations do not try to prevent cycling, and instead rely on small rounding errors to eventually move off the cycle.
Bland’s Rule:

1. Select the first column with positive coefficient in Z-row.
2. If there is a tie in Min-Ratio test, choose the first row within the tie.

Theorem: Following Bland’s Rule, the simplex method will always terminate in a finite number of steps.