Examples (i) and (ii) are sometimes referred to as *Leslie population models*, while (iii) is referred to in economics as a *Leontief model* of a multisectored economy.

Exercises

1. There are two jars initially full of pure water. Jar A is 0.5 liter and jar B is 0.25 liter. Water containing salt at a concentration of 100 g/liter is pumped into jar A at 0.5 liter/hr. Water flows from jar A to jar B at 0.5 liter/hr. Water flows out of jar B and down a drain at 0.5 liter/hr. Find the amounts of salt in each jar as a function of time t. Graph your solutions. (Figure 6.4.1 is applicable.)
2. There are two laboratory beakers. Beaker A contains 0.15 liter of pure water and beaker B contains 20 g of salt dissolved in 0.1 liter of water. Water containing salt at a concentration of 100 g/liter flows into beaker A at a rate of 0.3 liter/hr. Water flows from beaker A to beaker B at 0.3 liter/hr. Water flows from beaker B at 0.3 liter/hr and goes down the drain. Find the amount of salt in each beaker as a function of time, and graph your solution.
3. Water containing salt at a concentration of 1 lb/gal flows into a 2-gallon tank at 0.2 gal/min. Water flows out of the first tank and into a second 3-gallon tank at 0.2 gal/min. Both tanks are initially full of pure water. Pure water from a tap flows directly into the second tank at 0.1 gal/min. Water is piped out of the second tank and down a drain at 0.3 gal/min. Find the amount of salt in each tank as a function of time and graph your solution.
4. A tank initially contains 2 liters of water and 5 g of salt. Water containing salt at a concentration of 5 g/liter flows into this tank at 2 liters/hr. Water flows from this tank into a second tank at 2 liters/hr. The second tank contains one liter of fluid and initially contains 10 g of salt. Water evaporates from the second tank at 1 liter/hr and flows from the second tank and down a drain at 1 liter/hr. Find the amount of salt in each tank as a function of time, and graph the solutions.
5. There are two tanks. Tank A is a 100-gallon tank initially full of water containing salt at a concentration of 0.5 lb/gal. Tank B is a 200-gallon tank, initially full of water containing salt at a concentration of 0.1 lb/gal. Starting at time $t = 0$, water is pumped from tank A to tank B at 2 gal/min and from tank B to tank A at 2 gal/min. Find the amount of salt in each tank as a function of time, and graph the solutions.
6. There are two 100-gallon tanks full of water. Tank A contains salt at a concentration of 0.4 lb/gal, while tank B contains pure water. Pure water flows into tank A from an outside source at 2 gal/min. Water is pumped from tank A to tank B at 3 gal/min. Water evaporates from tank B at a rate of 2 gal/min. Water is also pumped from tank B to tank A at 1 gal/min. Find the amount of salt in each tank as a function of time, and graph your solution.
7. Two 2-liter jars are initially full of pure water. Water containing salt at a concentration of 10 g/liter is pumped into jar A at a rate of 2 liters/hr and into jar B at a rate of 2 liters/hr. Water is piped from jar A to jar B at 3 liters/hr and from jar B to jar A at 1 liter/hr. In addition, water flows from jar B down the drain at 4 liters/hr. Find the amount of salt in the jars as a function of time, and graph the solutions.
8. There are two tanks. Tank A contains 100 gallons of pure water while tank B contains 10 lb of salt dissolved in 200 gallons of water. Pure water enters tank A at 5 gal/min. Water is pumped from tank A to tank B at 8 gal/min and from tank B to tank A

at 3 gal/min. In addition, 5 gal/min of water is pumped out of tank B and sent out of the system. Find the amount of salt in each tank as a function of time, and graph the amounts.

9. Two large tanks initially each contain 100 gal of pure water. Water containing salt at a concentration of 0.5 lb/gal is pumped into tank A at 14 gal/min. Water is pumped from tank A to tank B at 9 gal/min. Water is pumped from tank B and sent down a drain at 6 gal/min. Find a differential equation for the amount of salt in each tank that is valid as long as the tanks are not full. You do not need to solve the differential equation. (*Note*: The volumes are not constant.)

10. Each of two large tanks in the desert initially contains 500 gallons of water with a salt concentration of 0.01 lb/gal. Water containing salt at a concentration of 0.2 lb/gal is pumped into tank A at a rate of 10 gal/hr. Water evaporates from tank A at a rate of 7 gal/hr and is pumped from tank A into tank B at 5 gal/hr. Water evaporates from tank B at a rate of 8 gal/hr. Water is also pumped out of tank B at 3 gal/hr. Write the differential equation for the amount of salt in each tank, which is valid until one of the tanks goes dry. You do not need to solve the differential equation.

11. There are three tanks each initially containing 100 gallons of pure water. Water containing salt at a concentration of 2 lb/gal flows into tank A at a rate of 5 gal/min. Water is pumped from tank A to tank B at 2 gal/min and from tank A to tank C at 3 gal/min. Water is pumped from tank C to tank B at 3 gal/min. Water is dumped out of tank B and then down a drain at a rate of 5 gal/min. Derive the differential equation for the amount of salt in each tank as a function of time. You need not actually solve the differential equation.

12. Three shallow ponds are out in the sun. Each is initially full of 5000 gallons of pure water. Water containing salt at a concentration of 0.2 lb/gal flows into pond A at a rate of 10 gal/hr. Water evaporates from pond A at 2 gal/hr. Water flows from pond A to pond B at 8 gal/hr. Water evaporates from pond B at 3 gal/hr. Water flows from pond B to pond C at 5 gal/hr. In pond C, water evaporates at 4 gal/hr and flows out at 1 gal/hr. Derive the differential equation for the amount of salt in each pond. You do not need to solve the differential equation.

13. Two tanks contain V_1 and V_2 gallons of water, respectively. Water containing salt at a concentration of δ lb/gal flows into the first tank at α gal/min, while β gal/min $(0 < \beta \le \alpha)$ of water is pumped from the first tank into the second. Finally, γ gal/min of water is pumped out of the second tank $(0 \le \gamma \le \beta)$. Assume that evaporation rates for the tanks are such that V_1, V_2 are constant.

   i) Set up the differential equation for the amount of salt in each tank.

   ii) Using elimination, find a differential equation for the amount of salt in the second tank.

   iii) Find the roots of the characteristic equation from part (ii).

   iv) Determine for what values of the parameters there will be equilibrium solutions and determine the equilibrium solution if there is one.

(*Population Models*) Exercises 14 through 19 deal with the following situation. A species is divided into m groups which we take to be age groups. Suppose that w is the size of a group at time t. For that group we assume

   i) There is a loss due to death which is proportional to group size, $-\delta w$, with $\delta > 0$. (This could also represent harvesting for species such as trees.)

   ii) Individuals "graduate" from one group to the next at a rate proportional to group size, $-gw$, with $g > 0$.

   iii) Fertile groups give rise to offspring at a rate proportional to group size, βw, with $\beta > 0$.

14. Suppose that a population consists of two groups: adults and children. Let x be the number of children and y the number of adults. Assume that children cannot have offspring. Explain why the model

$$\frac{dx}{dt} = -(\delta_1 + g)x + \beta y, \qquad x(t_0) = x_0 \ge 0$$
$$\frac{dy}{dt} = -\delta_2 y + gx, \qquad y(t_0) = y_0 \ge 0 \tag{20}$$

with δ_1, δ_2, g, β positive constants might be reasonable given assumptions (i) through (iii).

Exercises 15 through 17 illustrate the types of behavior possible for (20).

15. Suppose that (20) holds and $\delta_1 = g = 1$, $\delta_2 = 2$, and $\beta = 1$. Show that $\lim_{t\to\infty} x(t) = 0$, $\lim_{t\to\infty} y(t) = 0$, so that the species dies out.

16. Suppose that (20) holds and $\delta_1 = g = 1$, $\delta_2 = 2$, and $\beta = 4$.

a) Show that (20) has nonzero equilibrium solutions $(\bar{x}, \bar{y})$.

b) Show that every solution $x(t)$, $y(t)$ converges to one of these equilibrium solutions.

17. Suppose that (20) holds and $\delta_1 = g = 1$, $\delta_2 = 2$, and $\beta = 9$. Show that for initial conditions $x(0) > 0$, $y(0) > 0$ we have $\lim_{t\to\infty} x(t) = \infty$, $\lim_{t\to\infty} y(t) = \infty$, and $\lim_{t\to\infty} \dfrac{x(t)}{y(t)}$ exists and is finite.

18. Suppose that (20) holds. Show that

a) If $\delta_2(\delta_1 + g) - \beta g > 0$, then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$.

b) If $\delta_2(\delta_1 + g) - \beta g = 0$, then there are nonzero equilibriums and every solution of (20) converges to one of these equilibriums as $t \to \infty$.

c) If $\delta_2(\delta_1 + g) - \beta g < 0$, then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \infty$ for initial conditions $x(0) > 0$, $y(0) > 0$.

d) Give a biological interpretation of parts (a) through (c) in terms of the effect of the birth rate constant β.

19. Suppose that the population consists of three stages which we shall call larva (x), pupa (y), and adult (z). Only adults can produce larva.

a) Explain why a reasonable model might be

$$\begin{aligned} x' &= -(\delta_1 + g_1)x + \beta z \\ y' &= -(\delta_2 + g_2)y + g_1 x \\ z' &= -\delta_3 z + g_2 y \end{aligned} \tag{21}$$

with all constants positive and the initial conditions nonnegative.

b) Discuss what other assumptions would probably need to be made for (21) to be an accurate model.

20. (*Interest*) Two investment accounts are set up with \$1000 initially in account A and \$2000 initially in account B. Account A, the long-term account, earns 10% a year compounded daily. Account B earns 5% a year compounded daily. Deposits are made into B at the rate of \$10 a day. Every day the bank transfers money from B to A at an (annual) rate of 20% of the difference between B and \$2000. Set up the differential equations that model this situation. (Interest is first discussed in Exercise 2.9.15.)

6.5 Mechanical Systems

In this section we will discuss how some mechanical systems can be modeled by linear systems of differential equations. This is a continuation of Sections 2.13, 3.3, and 3.16. In order to avoid nonlinear problems, we shall consider point masses, connected by springs in a linear array, undergoing small oscillations. Larger three-dimensional arrays can be used to model many physical structures and mechanical devices. We consider two configurations.

A Horizontal Array of Springs and Masses

Suppose we have two point masses of mass m_1, m_2 and three springs of lengths L_1, L_2, L_3 and spring constants k_1, k_2, k_3. The springs and masses are arranged as in Fig. 6.5.1.

The left end of spring 1 and the right end of spring 3 are attached to immovable surfaces. The masses are in contact with a surface whose coefficient