MATH 616 HOMEWORK
DUE 4/2/18

(1) Show that $(\text{Image}\, A)^\perp \subseteq \text{Ker}(A^T)$.
(This completes the proof that $(\text{Image}\, A)^\perp = \text{Ker}(A^T)$)

(2) Prove that if $\text{Ker}\, A = 0$, then $\text{Ker}(A^T A) = 0$. Use this to conclude that if $\text{Ker}\, A = 0$, then $(A^T A)^{-1}$ exists.

(3) (a) Find the least squares solution of

$$\begin{aligned} x_1 + x_2 &= 4 \\ 2x_1 + x_2 &= -2 \\ x_1 - x_2 &= 1. \end{aligned}$$

(b) Use your answer to find the point on the plane spanned by $(1, 2, 1)$ and $(1, 1, -1)$ that is closest to $(4, -2, 1)$.

(4) Find the parabola $y = ax^2 + bx + c$ that best fits the four data points $(-1, 0), (0, 1), (1, 3), (2, 5)$.

(5) The goal of this problem is to understand why the method we've discussed is called the "least squares solution."
Use the same setup from problem (4) above.
(a) Explicitly write the desired equation in the form $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = (a, b, c)$, and $\mathbf{b} \in \mathbb{R}^4$.
(b) Using the standard inner product on $\mathbb{R}^4$ (given by the dot product), calculate the distance (squared) between $A\mathbf{x}$ and $\mathbf{b}$ for *arbitrary* $\mathbf{x} = (a, b, c)$. In other words, what is $|A\mathbf{x} - \mathbf{b}|^2$?
(c) Using 5b, explain why the solution to problem 4 is called the least squares solution.

(6) A square matrix $A \in \mathcal{M}_{n\times n}$ is said to be *orthogonal* if the linear map it represents preserves lengths and angles. Explicitly, viewing $A : \mathbb{R}^n \to \mathbb{R}^n$, we say A is orthogonal if:

$$\langle A\mathbf{x_1},\, A\mathbf{x_2}\rangle = \langle \mathbf{x_1},\, \mathbf{x_2}\rangle \text{ for all } \mathbf{x_1}, \mathbf{x_2} \in \mathbb{R}^n.$$

Prove: $A \in \mathcal{M}_{n\times n}$ is orthogonal if and only if $A^T A = A\, A^T = I$.
(This latter property is often used as the the definition of an orthogonal matrix.)

Date: March 26, 2018.