Math 616 - 1/29/18

Reminder (Last Semester)

V vector space (finite dimensional)

Linear map $T: V \to V$

Given basis B $\left(=\{\vec{v}_1, \dots, \vec{v}_n\}\right)$ of V, we can write Representation of $T: V \to V$ wrt B.

$$\begin{tikzcd} V \arrow[d, shift right, "{[\,]_B}"', "\cong"] & V \arrow[l, "T"'] \arrow[d, shift right, "{[\,]_B}"', "\cong"] \\ \mathbb{R}^n \arrow[u, shift right, "L_B"'] & \mathbb{R}^n \arrow[l, "T_{BB}"] \arrow[u, shift right, "L_B"'] \end{tikzcd}$$

$\boxed{T_{BB} \text{ is an } n\times n \text{ Matrix}}$

! Change of Basis: $T_{CC} = P_{CB} T_{BB} P_{BC} = P_{BC}^{-1} T_{BB} P$

Let $T: V \to V$ linear map. (λ is scalar)

Defn $\vec{v} \in V$ $(\vec{v} \neq \vec{0})$ is an eigenvector with eigenvalue λ if $T(\vec{v}) = \lambda \vec{v}$.

Example $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ is lin map $\mathbb{R}^2 \to \mathbb{R}^2$

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A\vec{v}_1 = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \vec{v}_1$ is evector w/ eigenvalue $\lambda = 3$.

$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A\vec{v}_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is evect w/ eval $\lambda = 4$.

In general, a diagonal matrix

$A = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$, viewed as a lin map $\mathbb{R}^n \to \mathbb{R}^n$,

has e-vectors $\vec{e}_i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, e-val λ_i.

Remark Are e-vects unique?

Suppose $T(\vec{v}) = \lambda\vec{v}$, then $T(a\vec{v}) = aT(\vec{v}) = a(\lambda\vec{v}) = \lambda(a\vec{v})$
& a is scalar; Hence $a\vec{v}$ is also evect w/ eval λ.

Ex $A = \begin{pmatrix} 4 & 1 \\ -8 & -5 \end{pmatrix}$ $\qquad \mathbb{R}^2 \xleftarrow{A} \mathbb{R}^2$, $A\vec{v} \leftarrow\!\shortmid \vec{v}$

② $A - \lambda I = \begin{pmatrix} 4-\lambda & 1 \\ -8 & -5-\lambda \end{pmatrix}$

$\det(A-\lambda I) = (4-\lambda)(-5-\lambda) + 8 = \lambda^2 + \lambda - 12 = (\lambda-3)(\lambda+4)$

Solving $\det(A-\lambda I) = 0$, $\Rightarrow \underbrace{\lambda = -4, 3}_{\text{2 eigenvalues}}$

③ $\lambda = -4$ $(A+4I)\vec{v} = \vec{0}$

$\begin{bmatrix} 8 & 1 \\ -8 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 8 & 1 \\ 0 & 0 \end{bmatrix}$ 1-dimensional e-space, evect $\vec{v}_1 = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$

$\lambda = 3$ Solve $(A - 3I)\vec{v} = \vec{0}$

$A - 3I = \begin{bmatrix} 1 & 1 \\ -8 & -8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ General sol'n $x = -t$, $y = t$

$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ evect w/ eval $\lambda = 3$.

④ Standard basis $\mathcal{E} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$. $A = T_{\mathcal{E}\mathcal{E}}$

$B = \{\vec{v}_1, \vec{v}_2\} = \left\{\begin{bmatrix} 1 \\ -8 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$. $T_{BB} = P_{B\mathcal{E}} T_{\mathcal{E}\mathcal{E}} P_{\mathcal{E}B}$

$$\boxed{\begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -8 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 1 \\ -8 & -5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -8 & 1 \end{bmatrix}}$$

Note $\boxed{\text{If } D = P^{-1}AP}$

$\Rightarrow PD = \cancel{P}\cancel{P^{-1}}AP$

$PDP^{-1} = AP \cdot P^{-1}$

$\boxed{PDP^{-1} = A}$

Ex 2 Diagonalize $A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

Characteristic Polynomial: $\det(A - \lambda I)$

$$\begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 12\lambda - \lambda^2 - \lambda^3 = -\lambda(\lambda^2 + \lambda - 12) = -\lambda(\lambda+4)(\lambda-3)$$

$\Rightarrow$ evals $\lambda = -4, 0, 3$

$\underline{\lambda_1 = -4}$ $\begin{bmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

evect $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ w/ eval $\lambda_1 = -4$.

$\underline{\lambda_2 = 0}$ $\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1/13 \\ 0 & 1 & 6/13 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x = -\frac{1}{13}t$, $y = -6/13\,t$, $z = t$

$\vec{v}_2 = \begin{bmatrix} -1 \\ -6 \\ 13 \end{bmatrix}$, $\lambda_2 = 0$.

$\underline{\lambda_3 = 3}$ $\begin{bmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$ w/ $\lambda_3 = 3$

Letting $D = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be new basis; $A = \_[\text{Diag}]\_$

$T_{EE} = P_{ED} T_{DD} P_{DE}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -2 \\ 2 & -6 & -3 \\ 1 & 13 & 2 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -1 & -2 \\ 2 & -6 & -3 \\ 1 & 13 & 2 \end{bmatrix}^{-1}$$

General If $A \in M_{n \times n}$ has n distinct real e-vals $\lambda_1, \dots, \lambda_n$ with associated e-vectors $\underbrace{\vec{v}_1, \dots, \vec{v}_n}_{\text{Basis } D}$

Then

$$A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}^{-1}$$

Ex $A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$ ① Find e-values

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 3 = \lambda^2 - 2\lambda + 4 \overset{\text{Solve}}{=} 0$$

$$\lambda = \frac{2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{3}\,i}{2}$$

Complex Roots $\lambda = 1 \pm \sqrt{3}\,i$

$\lambda_1 = 1+\sqrt{3}i$ $\quad \begin{bmatrix} -\sqrt{3}i & 3 \\ -1 & -\sqrt{3}i \end{bmatrix} \rightsquigarrow \begin{bmatrix} -\sqrt{3} & -3i \\ -1 & -\sqrt{3}i \end{bmatrix}$

$\rightsquigarrow \begin{bmatrix} -1 & -\sqrt{3}i \\ -1 & -\sqrt{3}i \end{bmatrix} \sim \begin{bmatrix} 1 & \sqrt{3}i \\ 0 & 0 \end{bmatrix}$

<u>Note</u> $i \cdot i = -1$

$i = \frac{-1}{i}$

$\frac{1}{i} = -i$

$\{ x + (\sqrt{3}i)y = 0$

<u>E-vect</u> $\vec{v}_1 = \begin{bmatrix} -\sqrt{3}i \\ 1 \end{bmatrix}$

$\lambda_2 = 1 - \sqrt{3}i$ $\quad \begin{bmatrix} +\sqrt{3}i & 3 \\ -1 & +\sqrt{3}i \end{bmatrix}$

$\boxed{\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}i & \sqrt{3}i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+\sqrt{3}i & 0 \\ 0 & 1-\sqrt{3}i \end{bmatrix} \begin{bmatrix} -\sqrt{3}i & \sqrt{3}i \\ 1 & 1 \end{bmatrix}^{-1}}$ $\quad \begin{bmatrix} 1 & -\sqrt{3}i \\ 0 & 0 \end{bmatrix}$

$\{ x - (\sqrt{3}i)y = 0$

<u>E-vect</u> $\vec{v}_2 = \begin{bmatrix} +\sqrt{3}i \\ 1 \end{bmatrix}$