Linear Programming, Lecture 4

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Standard Form

For an LP, "Standard Form" is *usually* defined as

$$\text{Maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b}$$

or

$$\text{Maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}.$$

or

$$\text{Maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}.$$

Simplex Method

To run the simplex method, we start from a Linear Program (LP) in the following *standard simplex form.*

$$
\begin{aligned}
\text{Max} \quad & z \\
\text{s.t.} \quad (-z) + a_{01}x_1 + \cdots + a_{0n}x_n &= b_0 \\
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
&\vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \\
x_i &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Max} \quad & z \\
\text{s.t.} \quad (-z) + a_{01}x_1 + \cdots + a_{0n}x_n &= b_0 \\
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
&\vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \\
x_i &\geq 0
\end{aligned}
$$

To be in standard simplex form:

1. All decision variables x_i (except $-z$) are non-negative.
2. All other constraints are equalities.
3. The RHS (except for the "cost row" or "z-row") is non-negative.
4. For each row i, there is a column equal to e_i (a 1 in row i, and 0 in all other rows).

Remarks

- There are multiple conventions as to what constitutes "Standard Form." They are all different, but more or less equivalent in terms of requirements.
- Given an LP in the form: Max z, subject to inequalities all of the form $\sum a_i x_i \leq b$, one only needs to introduce slack variables to obtain starting standard form for Simplex Method.
- Today, we will learn techniques for more complicated LPs.

Example 1

Is the following LP in standard simplex form?
Maximize z, subject to $x_1, x_2, x_3, s_1, s_2 \geq 0$ and the equalities:

$-z$	x_1	x_2	x_3	s_1	s_2	$=$	RHS
1	1	9	1	0	0		0
0	1	2	3	1	0		9
0	3	2	2	0	1		15

- Non-negative decision variables? $\checkmark$
- Equalities for constrains? $\checkmark$
- Non-negative RHS entries? $\checkmark$
- Columns e_i? $\checkmark$ $-z, s_1, s_2$

Example 2

Is the following LP in standard simplex form?
Maximize z, subject to $x_1, x_2, x_3, s_1, s_2 \geq 0$ and the equalities:

$-z$	x_1	x_2	x_3	s_1	s_2	$=$	RHS
1	1	9	1	0	0		0
0	1	2	3	-1	0		9
0	3	2	2	0	1		15

- Non-negative decision variables? ✓
- Equalities for constrains? ✓
- Non-negative RHS entries? ✓
- Columns e_i? ✗ No column $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$

Potential complications

1. Minimizing instead of maximizing.
2. Decision variables allowed to take negative values.
3. Inequalities of form $\sum a_i x_i \geq b$.

Maximize/Minimize

$$\boxed{\text{Maximizing } f(x_1, \ldots, x_n) \quad \Leftrightarrow \quad \text{Minimizing } -f(x_1, \ldots, x_n)}$$

Notes:

- To change between max/min, just multiply all coefficients by -1.
- In practice, some implementations of simplex method assume you are maximizing, and some assume you are minimizing. Max vs Min is a minor detail.
- When doing simplex method by hand, you may simply keep cost row the same as when maximizing, but perform pivots on columns with a *negative* entry in the cost row.

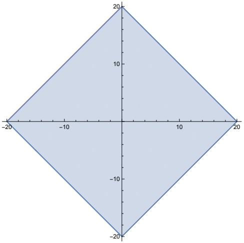
Decision variables taking negative values

Bounded below: Suppose we have a variable $x \geq -20$.
Then: Substitute for new variable $x = \hat{x} - 20$, or $\hat{x} = x + 20$, with $\hat{x} \geq 0$.
Note: For constraint $x \geq 20$, we may introduce surplus variable, or we may use substitution $x = \hat{x} + 20$, with $\hat{x} \geq 0$.

Unbounded: Suppose we have unbounded variable w.
Then: Use substitution $w = w^+ - w^-$, with $w^+, w^- \geq 0$.
Note: When more than one decision variable is unrestricted, a single variable x^- can be used for all of them, with the interpretation that it is the most negative of all the decision variables.

Put LP in Simplex Form to start Simplex Method

$$
\begin{array}{ll}
\text{Min} & x + y \\
\text{s.t.} & x + y \leq 20 \\
& x + y \geq -20 \\
& x - y \leq 20 \\
& x - y \geq -20 \\
& x, y \text{ unrestricted}
\end{array}
$$



Question: Will the LP have a unique solution?

Inequalities:

$$-20 \leq x + y \leq 20$$
$$-20 \leq x - y \leq 20$$

Substitutions:

$$x = x^+ - x^- \text{ and } y = y^+ - y^-$$
$$x^+, x^-, y^+, y^- \geq 0$$

New inequalities:

$$-20 \leq x^+ - x^- + y^+ - y^- \leq 20$$
$$-20 \leq x^+ - x^- - y^+ + y^- \leq 20$$

$$
\begin{aligned}
\text{Min} \quad & x^+ - x^- + y^+ - y^- \\
\text{s.t.} \quad & x^+ - x^- + y^+ - y^- \leq 20 \\
& x^+ - x^- + y^+ - y^- \geq -20 \\
& x^+ - x^- - y^+ + y^- \leq 20 \\
& x^+ - x^- - y^+ + y^- \geq -20 \\
& \quad x^+, x^-, y^+, y^- \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Max} \quad z = -x^{+} + x^{-} - y^{-} + y^{-} & \\
\text{s.t.} \qquad x^{+} - x^{-} + y^{+} - y^{-} &\leq 20 \\
x^{+} - x^{-} + y^{+} - y^{-} &\geq -20 \\
x^{+} - x^{-} - y^{+} + y^{-} &\leq 20 \\
x^{+} - x^{-} - y^{+} + y^{-} &\geq -20 \\
x^{+}, x^{-}, y^{+}, y^{-} &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Max } \; z &= -x^+ + x^- - y^- + y^- \\
\text{s.t.} \quad & x^+ - x^- + y^+ - y^- + s_1 = 20 \\
& x^+ - x^- + y^+ - y^- - s_2 = -20 \\
& x^+ - x^- - y^+ + y^- + s_3 = 20 \\
& x^+ - x^- - y^+ + y^- - s_4 = -20 \\
& x^+, x^-, y^+, y^- \geq 0
\end{aligned}
$$

Maximize z subject to non-negativity constraints and:

$(-z)$	x^+	x^-	y^+	y^-	s_1	s_2	s_3	s_4	RHS
1	-1	1	-1	1	0	0	0	0	0
0	1	-1	1	-1	1	0	0	0	20
0	1	-1	1	-1	0	-1	0	0	-20
0	1	-1	-1	1	0	0	1	0	20
0	1	-1	-1	1	0	0	0	-1	-20

$(-z)$	x^+	x^-	y^+	y^-	s_1	s_2	s_3	s_4	RHS
1	-1	1	-1	1	0	0	0	0	0
0	1	-1	1	-1	1	0	0	0	20
0	-1	1	-1	1	0	1	0	0	20
0	1	-1	-1	1	0	0	1	0	20
0	-1	1	1	-1	0	0	0	1	20

Alternate Options:

- Since two variables are unbounded, we could substitute $x = x^+ - t$ and $y = y^+ - t$, where $x^+, y^+, t \geq 0$. Here, $t = \max\{-x, -y, 0\}$.
- Inspecting the inequalities, we observe that $x, y \geq -20$. Hence, we could use substitutions $x = \hat{x} - 20, \quad y = \hat{y} - 20, \quad \hat{x}, \hat{y} \geq 0$.

$\geq$ Inequalities

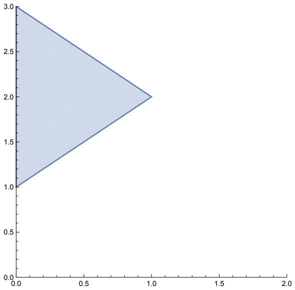
Given: $\sum_j a_{ij}x_j \geq b_i$, with $b_i > 0$
Introduce: surplus variable $s \geq 0$ to form

$$\sum_j a_{ij}x_j - s_i = b_i.$$

Problem: Neither $\sum_j a_{ij}x_j - s_i = b_i$ nor $\sum_j -a_{ij}x_j + s_i = -b_i$ give something in simplex form.
Solution: Introduce *artificial variable(s)* $\alpha_i \geq 0$, $\sum_j a_{ij}x_j - s_i + \alpha_i = b_i$. Then, solve the related LP with objective function $\sum \alpha_i$, and same constraints. This produces an initial Basic Feasible Solution and immediately translates to the LP in simplex form.

Example

$$\begin{aligned} \text{Max} \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 3 \\ & -x_1 + x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Phase I auxiliary LP:

$$\begin{array}{lrl} \text{Min} & x_5 & \\ \text{s.t.} & x_1 + x_2 + x_3 & = 3 \\ & -x_1 + x_2 - x_4 + x_5 & = 1 \\ & x_1, \ldots x_5 & \geq 0 \end{array}$$

$-w$	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	0	0	-1	0
0	1	1	1	0	0	3
0	-1	1	0	-1	1	1

$-w$	x_1	x_2	x_3	x_4	x_5	RHS
1	-1	1	0	-1	0	1
0	1	1	1	0	0	3
0	-1	1	0	-1	1	1

Solution to Phase I auxiliary LP:

$-w$	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	0	0	-1	0
0	2	0	1	1	-1	2
0	-1	1	0	-1	1	1

Solution of $x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 0$ gives $x_5 = 0$.
These values $(x_1, \ldots, x_4)$ satisfy the original LPs constraints, and they form our initial Basic Feasible Solution!

Phase II, Solving initial LP:

$-z$	x_1	x_2	x_3	x_4	RHS
1	2	1	0	0	0
0	2	0	1	1	2
0	-1	1	0	-1	1

ERO's to isolate x_2, x_3, then in simplex form.

Suppose that when performing the simplex method, you obtain column with positive number in objective row, and non-positive numbers in rest of column. Then, the feasible region is unbounded, and **a solution does not exist.**

Example:

$-z$	x_1	x_2	x_3	x_4	RHS
1	2	0	0	0	0
0	-2	0	1	1	2
0	-1	1	0	-1	1

For *basic solution*, we let $x_1 = 0$. But, we can let x_1 be any positive number, and we obtain a *better* feasible solution.

Cycling

When several iterations of the simplex method do not improve the current objective value, this is called **stalling.**

When, after several iterations, the simplex method returns a previous tableau, this is called **cycling.**

In general, stalling and cycling can occur. Some implementations of the simplex method include special provisions to prevent cycling; other implementations do not try to prevent cycling, and instead rely on small rounding errors to eventually move off the cycle.

Bland's Rule:

1. Select the first column with positive coefficient in Z-row.
2. If there is a tie in Min-Ratio test, choose the first row within the tie.

Theorem: Following Bland's Rule, the simplex method will always terminate in a finite number of steps.