NOTES FOR LINEAR ALGEBRA

CORBETT REDDEN
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1. Coordinates

Definition 1.1. Let $C = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a collection of vectors in a vector space V. We say that C

- is **linearly independent** if $(r_1\mathbf{v}_1 + \cdots r_n\mathbf{v}_n = \mathbf{0}) \Rightarrow (r_1 = \cdots = r_n = 0)$;
- **spans** V if for every $\mathbf{v} \in V$, exists r_i satisfying $r_1\mathbf{v}_1 + \cdots + r_n\mathbf{v}_n = \mathbf{v}$;
- is a **basis** for V if it is linearly independent and spans V.

Theorem 1.2. *Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for the vector space V. Any vector $\mathbf{v} \in V$ can be expressed uniquely as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$. In other words, there is a unique solution $(r_1, \dots, r_n)$ to the equation*

$$r_1\mathbf{v}_1 + \cdots + r_n\mathbf{v}_n = \mathbf{v}.$$

Definition 1.3. Suppose $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for the vector space V. Then, for a vector $\mathbf{v} \in V$, we say that the **coordinates** (or coordinate vector) of $\mathbf{v}$ with respect to the basis B is the unique vector $(r_1, \dots, r_n) \in \mathbb{R}^n$ such that $\mathbf{v} = \sum_i r_i\mathbf{v}_i$. We use the notation

$$[\mathbf{v}]_B = [r_1\mathbf{v}_1 + \cdots + r_n\mathbf{v}_n]_{\mathcal{E}} = (r_1, \dots, r_n) = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}.$$

Remark 1.4. The class textbook uses the notation $\mathrm{Rep}_B(\mathbf{v})$ instead of $[\mathbf{v}]_B$, and calls it the representation of $\mathbf{v}$ with respect to the basis B.

Example 1.5. $\mathbb{R}^n$ has the canonical basis $\mathcal{E} = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ where $\mathbf{e}_i = (0, \dots, 1, \dots, 0)$, and

$$[(x_1, \dots, x_n)]_{\mathcal{E}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Example 1.6. The vector space $\mathcal{P}_n$ has a standard basis $\{1, x, \dots, x^n\}$, and

$$[\sum_i a_i x^i]_B = (a_0, \dots, a_n).$$

Example 1.7. Let $B = \{(1,2), (3,1)\}$ be a basis for $\mathbb{R}^2$. Then, to find the coordinates of an arbitary vector $(a, b) \in \mathbb{R}^2$ with respect to B, we solve the equation

$$r_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + r_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 3 & a \\ 2 & 1 & b \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -\frac{1}{5}a + \frac{3}{5}b \\ 0 & 1 & \frac{2}{5}a - \frac{1}{5}b \end{bmatrix}$$

Therefore,

$$[(a, b)]_B = \begin{bmatrix} -\frac{1}{5}a + \frac{3}{5}b \\ \frac{2}{5}a - \frac{1}{5}b \end{bmatrix}.$$

More concretely,

$$[(5, 5)]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Note: the *order* of the vectors in the basis matters! Swapping the order will swap the corresponding columns in the coordinate vector.

Example 1.8. Consider the subspace V of $\mathcal{M}_{2\times 2}$ with the basis

$$B = \left\{ \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

Then, the coordinate vector $(5, -2) \in \mathbb{R}^2$ represents the matrix

$$5 \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix} - 2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -7 & -2 \\ 8 & 0 \end{bmatrix}$$

relative to the basis B.

To find the coordinates of $\begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$ relative to B, we solve

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and conclude that the coordinate vector is $(-2, 1) \in \mathbb{R}^2$.

2. Linear maps

The previous examples are all examples of *maps between vector spaces.* Given a finite-dimensional vector space V with basis B, we have a function (or mapping) that associates to any vector $\mathbf{v} \in V$ a vector in $\mathbb{R}^n$:

$$\mathbb{R}^n \xleftarrow{[]_B} V$$
$$[\mathbf{v}]_B \longleftarrow\!\shortmid \mathbf{v}$$

More generally, whenever we have collection of vectors $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ (not necessarily a basis), we can define a linear map given by taking *linear combinations*:

$$V \xleftarrow{L_C} \mathbb{R}^n$$
$$\sum_i r_i \mathbf{v}_i \longleftarrow\!\shortmid (r_1, \dots, r_n)$$

Remark 2.1. The book (and probably all of your previous textbooks) would usually write the above as $[]_B : V \to \mathbb{R}^n$ and $L_B : \mathbb{R}^n \to B$ which are read left to right. We will use the "right to left" notation. While it is a little confusing at first, it will be much more convenient later in the course when encountering function composition and matrix multiplication.

Definition 2.2. Let V and W be vector spaces. A **function** T from V to W, written $T : V \to W$ or $W \xleftarrow{T} V$, is a rule that assigns to each vector $v \in V$ a unique vector $T(v) \in W$.

Vocabulary: In addition to the word *function*, and the words *transformation* and *map* or *mapping* are also common; all have the same meaning. Given a function $W \xleftarrow{T} V$,

- V is called the *domain* and W is the *target space* or *codomain*.
- If $\mathbf{w} = T(\mathbf{v})$, then $\mathbf{w}$ is the *image of* $\mathbf{v}$ under T.
- The set of all images is called the *image* or *range* of T. The range may be a part of W or all of W.

Example 2.3. The function $f(x) = x^2$ has domain and target space $\mathbb{R}$.

A curve in the plane is a function $\mathbb{R}^2 \leftarrow \mathbb{R}$, and a curve in $\mathbb{R}^3$ is a function $\mathbb{R}^3 \leftarrow \mathbb{R}$. The domain is $\mathbb{R}$ in both cases, and the target space is $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

A vector field on the plane is a map $\mathbb{R}^2 \leftarrow \mathbb{R}^2$. The domain and target space are both $\mathbb{R}^2$.

Note that none of the above examples are assumed to be linear. The notions of domain/range/target apply to functions in general and do not rely on vector space structures.

Definition 2.4. A function $W \xleftarrow{T} V$ between vector spaces is **linear** if for all $\mathbf{u}, \mathbf{v} \in V$ and $r \in \mathbb{R}$,

$$T(r\mathbf{v}) \;=\; r\,T(\mathbf{u}) \qquad \text{and} \qquad T(\mathbf{u}+\mathbf{v}) \;=\; T(\mathbf{u}) + T(\mathbf{v}).$$

Lemma 2.5. *If $W \xleftarrow{T} V$ is linear, then for all $\mathbf{u}, \mathbf{v} \in V$ and $a, b \in \mathbb{R}$:*

$$(a)\;\; T(\mathbf{0}) \;=\; \mathbf{0} \qquad (b)\;\; T(-\mathbf{v}) \;=\; -T(\mathbf{v}) \qquad (c)\;\; T(a\mathbf{u}+b\mathbf{v}) \;=\; a\,T(\mathbf{u}) + b\,T(\mathbf{v}).$$

and (c) extends to general linear combinations: $T\left(\sum a_i \mathbf{v}_i\right) \;=\; \sum a_i\, T(\mathbf{v}_i)$.

Proof.

$$T(\mathbf{0}_V) = T(0\mathbf{v}) = 0T(\mathbf{v}) = \mathbf{0}_W,$$
$$T(-\mathbf{v}) = T((-1)\mathbf{v})) = (-1)T(\mathbf{v}) = -T(\mathbf{v}),$$
$$T(a\mathbf{u}+b\mathbf{v}) = T(a\mathbf{u}) + T(b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v}).$$

□

Remark 2.6. The above lemma shows that T linear implies $T(r\mathbf{u}+s\mathbf{v}) = rT(\mathbf{u}) + sT(\mathbf{v})$. The converse is also true, as demonstrated by setting $r = 1, s = 1$ or $s = 0$. Therefore, being linear is equivalent to

$$T(r\mathbf{u}+s\mathbf{v}) = rT(\mathbf{u}) + sT(\mathbf{v})$$

being satisfied for all $r, s \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in V$.

Example 2.7. Matrix multiplication defines linear maps. Let $A \in \mathcal{M}_{m\times n}$ be an $m \times n$ matrix. Then, A defines a linear map

$$\mathbb{R}^m \xleftarrow{A} \mathbb{R}^n$$
$$A\mathbf{x} \longleftarrow\!\shortmid\, \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Note that writing our function as moving right to left makes subcripts work out nicely. Vectors in $\mathbb{R}^a$ are written as $a \times 1$ matrices, and we have that $A_{m\times n}$ inputs vectors in $\mathbb{R}^n$ and outputs vectors in $\mathbb{R}^m$.

Example 2.8. The derivative is a linear map $C^{k-1}(\mathbb{R}) \overset{\frac{d}{dx}}{\leftarrow} C^k(\mathbb{R})$,, where $C^k(\mathbb{R})$ is the set of functions $\mathbb{R} \to \mathbb{R}$ that are continuous and whose first k derivatives are also continuous. This follows from standard properties of derivatives, as

$$\frac{d}{dx}(rf+sg) = \frac{d}{dx}(rf) + \frac{d}{dx}(sg) = r\frac{df}{dx} + s\frac{dg}{dx}.$$

Example 2.9. The linear map $\mathcal{P}_3 \xleftarrow{T} \mathcal{P}_2$ given by $T(p) = (x+1)p$ is linear. Check:

$$\begin{aligned} T(rp_1 + sp_2) &= (x+1)(rp_1+sp_2) = r(x+1)p_1 + s(x+1)p_2 \\ &= rT(p_1) + sT(p_2). \end{aligned}$$

Example 2.10. Given a basis B of V, the "coordinates" are really a linear map $\mathbb{R}^n \leftarrow V$. Checking this is linear is a homework assignment.

Lemma 2.11. *Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for the vector space V. A linear transformation $W \xleftarrow{T} V$ is determined by the values $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$; i.e.*

(a) *If we know $T(\mathbf{v}_i)$ for all i, we can calculate $T(\mathbf{v})$ for any vector $\mathbf{v} \in V$.*
(b) *If $W \xleftarrow{S} V$ is a linear map so that $S(\mathbf{v}_i) = T(\mathbf{v}_i)$ on each basis vector $\mathbf{v}_i$, then $S(\mathbf{v}) = T(\mathbf{v})$ for all vectors $\mathbf{v}$ in V.*

Proof. Given a basis B of V, any vector $\mathbf{v} \in V$ is uniquely written as $\mathbf{v} = \sum_i r_i \mathbf{v}_i$. If T is a linear map, then

$$T(\mathbf{v}) = T(\sum_i r_i \mathbf{v}_i) = \sum_i r_i T(\mathbf{v}_i),$$

so T is completely determined by its values on the basis vectors. Similarly, if S is another linear map which agrees with T on basis vectors, then

$$S(\mathbf{v}) = S(\sum_i r_i \mathbf{v}_i) = \sum_i r_i S(\mathbf{v}_i) = \sum_i r_i T(\mathbf{v}_i) = T(\mathbf{v}).$$

□

3. Kernel, Image, and Isomorphisms

Definition 3.1. Let $W \xleftarrow{T} V$ be any linear map. The kernel (or null space, nullity) and image (or range) are subspaces $\mathrm{Ker}(T) \subseteq V$ and $\mathrm{Image}(T) \subseteq W$ defined as follows:

- $\mathrm{Ker}(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\} \subseteq V$,
- $\mathrm{Image}(T) = \{T(\mathbf{v}) \in W \mid \mathbf{v} \in V\} \subseteq W$.

Definition 3.2. A linear map $W \xleftarrow{T} V$ is an *isomorphism* if there exists an inverse $W \xrightarrow{T^{-1}} V$ satisfying

$$T^{-1} \circ T = \mathrm{id}_V, \quad T \circ T^{-1} = \mathrm{id}_W .$$

We say that $V \cong W$, or V is isomorphic to W, if there exists an isomorphism between the two vector spaces.

Remark 3.3. We can conveniently use the commutative diagram

$$W \underset{T}{\overset{T^{-1}}{\rightleftarrows}} V \quad (\cong)$$

to encode this visually. The above diagram indicates that completing a "full loop" maps to the same element you start with.

Proposition 3.4. *A linear map $W \xleftarrow{T} V$ is an isomorphism if and only if* $\mathrm{Ker}(T) = \mathbf{0}$ *and* $\mathrm{Image}(T) = W$.

Example 3.5. Let $A = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be any collection of vectors in V. This defines a linear map

$$V \xleftarrow{L_A} \mathbb{R}^n$$
$$\sum_i r_i \mathbf{v}_i \longleftarrow\!\shortmid (r_1, \dots, r_n)$$

given by taking linear combinations of the vectors $\mathbf{v}_i$. It is an instructive exercise to show the following:

- $\mathrm{Ker}(L_A) = \mathbf{0}$ if and only if $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent.
- $\mathrm{Image}(L_A) = W$ if and only if $\mathrm{span}(\mathbf{v}_1, \dots, \mathbf{v}_n) = W$.
- L_A is an isomorphism if and only if $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for V.

If L_A is an isomorphism, the inverse is the coordinate map $[\,]_A$.

4. Coordinates of a linear map

Special Case: Let $A \in \mathcal{M}_{m\times n}$ be an $m \times n$ matrix, which is equivalent to a linear map

$$\mathbb{R}^m \xleftarrow{A} \mathbb{R}^n$$
$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longleftarrow\!\shortmid \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

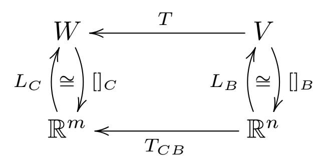
To understand what these numbers a_{ij} mean, let's see where A maps basis vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Tracing through matrix multiplication, we see that $A(\mathbf{e}_j)$ is the j-th *column* of the matrix A. In other words,

$$A(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad A(\mathbf{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \dots, \quad A(\mathbf{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

In other words, the columns of the matrix tell you the image of each basis vector in $\mathbb{R}^n$. There are n columns because there are n basis vectors in $\mathbb{R}^n$. There are m rows because each vector in $\mathbb{R}^m$ is described via the m basis vectors.

One we understand how matrix multiplication determines linear maps between the Euclidean vector spaces $\mathbb{R}^i$, we can use coordinates to better understand arbitrary linear maps in terms of matrix multiplication.

General Case: Let $W \xleftarrow{T} V$ be a linear map, and let B be a basis for V and C a basis for W. This gives the following commutative diagram and induces a map we call T_{CB}.



This map $\mathbb{R}^m \xleftarrow{T_{CB}} \mathbb{R}^n$ is simply multiplication by a $m \times n$ matrix, which we also denote by T_{CB}. By definition, this linear map must satisfy

$$T_{CB}[\mathbf{v}]_B = [T(\mathbf{v})]_C.$$

In other words, you can calculate the C-coordinates of $T(\mathbf{v})$ by multiplying the B-coordinates of $\mathbf{v}$ by T_{CB}. To construct the matrix T_{CB}, we just have to see where the vectors $\mathbf{e}_i$ are mapped. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be our basis. Tracing through the definitions will give us the formula

$$T_{CB} = \begin{bmatrix} [T(\mathbf{v}_1)]_C & [T(\mathbf{v}_2)]_C & \ldots & [T(\mathbf{v}_n)]_C \end{bmatrix}.$$

Here, the elements $[T(\mathbf{v}_j)]_C$ are considered to be columns in our matrix with dimensions $\dim W \times \dim V$.

Remark 4.1. The textbook uses the notation $\mathrm{Rep}_{B,C}(T)$ for what we call T_{CB}. It is essentially the same thing, except we reverse the order of the subscripts B, C. Remember that in this notation, the *input* is always occurs on the *right*, and the *output* is always on the *left*.

Example 4.2. Suppose that $B = \{(1,1), (1,-1)\}$ is a basis for $\mathbb{R}^2$ and $C = \{1, 1+x, 1+x+x^2\}$ is a basis for $\mathcal{P}_2$. Let $\mathcal{P}_2 \xleftarrow{T} \mathbb{R}^2$ be a linear map satisfying

$$T(1,1) = 3(1) + 2(1+x) + 0(1+x+x^2), \quad T(1,-1) = 0(1) + 0(1+x) - 5(1+x+x^2).$$

Then, the matrix representation of T with respect to the bases B and C is given by

$$T_{CB} = \begin{bmatrix} [T(1,1)]_C & [T(1,-1))]_C \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 0 & -5 \end{bmatrix}.$$

Suppose you want to know what $T(5,-1)$ equals. First, we find the B-coordinates (by solving an equation to show that)

$$2(1,1) + 3(1,-1) = (5,-1) \quad \Rightarrow \quad [(5,-1)]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Then, we can find the C-coordinates of $T(5,-1)$ by

$$[T(5,-1)]_C = T_{CB}[(5,-1)]_B = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -15 \end{bmatrix}.$$

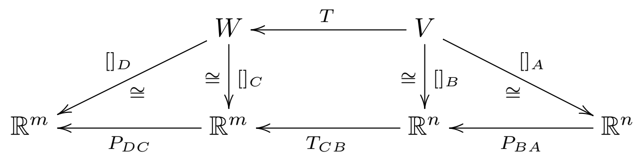
Therefore,

$$T(5,-1) = 6(1) + 4(1+x) - 15(1+x+x^2).$$

5. Change of basis formulas

Suppose that we the representation of a linear map with respect to one set of coordinates, but we want it with respect to different coordinates. What do we do? Is there a systematic way to deal with this type of situation? **YES!**

Suppose A, B are two bases for V, and C, D are two bases for W, and $W \xleftarrow{T} V$ is linear. Then, we have



The matrices P_{DC} and P_{BA} are called *change of basis matrices* and are determined by representing the identity map with respect to two different bases. This gives the general formula

$$P_{BA} = \begin{bmatrix} [\mathbf{v}_1^A]_B & [\mathbf{v}_2^A]_B & \dots & [\mathbf{v}_n^A]_B \end{bmatrix}$$

where $A = \{\mathbf{v}_1^A, \dots, \mathbf{v}_n^A\}$. These give the following important change of basis formulas:

$$\boxed{\begin{aligned} &T_{DA} = P_{DC} T_{CB} P_{BA}, \\ &P_{AB} = P_{BA}^{-1}. \end{aligned}}$$

Composition of linear functions is also compatible with matrix multiplication. Specifically, suppose that $(S \circ T)$ is a composition of linear maps. Then

$$\boxed{(S \circ T)_{DB} = S_{DC} T_{CB}.}$$

Example 5.1. Consider Example 4.2 above. We had $B = \{(1, 1), (1, -1)\}$, $C = \{1, 1 + x, 1 + x + x^2\}$, and $\mathcal{P}_2 \xleftarrow{T} \mathbb{R}^2$ with matrix

$$T_{CB} = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 0 & -5 \end{bmatrix}.$$

What if we want the matrix of T with respect to the standard bases $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $D = \{1, x, x^2\}$? We can just compute the change of basis matrices.

$$P_{\mathcal{E}B} = \begin{bmatrix} [(1,1)]_{\mathcal{E}} & [(1,-1)]_{\mathcal{E}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$P_{DC} = \begin{bmatrix} [1]_D & [1+x]_D & [1+x+x^2]_D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Using our change of basis formulas, we have

$$T_{D\mathcal{E}} = P_{DC} T_{CB} P_{B\mathcal{E}} = P_{DC} T_{CB} P_{\mathcal{E}B}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 10 \\ -3 & 7 \\ -5 & 5 \end{bmatrix}$$

6. Calculating Kernel and Image

6.1. Finding Kernel and Image of a matrix.

Let $A \in \mathcal{M}_{m \times n}$ be a matrix. The kernel of A (or nullspace)

$$\text{Ker}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

is just the solution set to the system of homogeneous equations associated to the matrix A.
The Image (or range or column space) is

$$\operatorname{Image} A = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\} = \{\mathbf{y} \in \mathbb{R}^m \mid \exists \mathbf{x} \text{ satisfying } \mathbf{y} = A\mathbf{x}\},$$

and it is easy to show this equals the span of the columns of A.

To find a basis for the kernel and image of a matrix A:

(1) Using row reduction, put A in reduced row echelon form (RREF).
(2) Once A is in RREF, write down the set of solutions to the linear system. You will get a basis vector for each column with a free variable (however the basis vector is *not* the column vector!).
(3) Image A has a basis given by the columns of A which, after row reduction, contain a pivot.

You will always have $\dim \operatorname{Ker} A =$ the number of free variables, and $\dim \operatorname{Image} A =$ number of pivots.

6.2. **Finding Kernel and Image of a general linear map between finite-dimensional vector spaces.**

$$\operatorname{Ker} T = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\} \subseteq V, \quad \operatorname{Image} T = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\} \subseteq W.$$

(1) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map given by $T(\mathbf{x}) = A\mathbf{x}$, then $\operatorname{Ker} T = \operatorname{Ker} A$, and $\operatorname{Image} T = \operatorname{Image} A$. You are done.
(2) For a general linear map $T : V \to W$, choose bases B, C of V, W. Determine the matrix T_{CB}, which represents the linear map relative to the bases B and C.
(3) Find a basis for kernel/image of the matrix T_{CB}.
(4) For each vector in the basis of $\operatorname{Ker} T_{CB} \subset \mathbb{R}^n$, map it to V by L_B. Here, $L_B(r_1, \ldots, r_n) = r_1\mathbf{v}_1 + \cdots r_n\mathbf{v}_n$ where $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.
(5) For each vector in the basis of $\operatorname{Image} T_{CB} \subset \mathbb{R}^m$, map it to W by L_C.
(6) Note that $\dim \operatorname{Ker} T = \dim \operatorname{Ker} T_{CB} = \#$ free variables, and $\dim \operatorname{Image} T = \dim \operatorname{Image} T_{CB} = \#$ pivots.

Remark 6.1. Equivalently, you can set up the equations

$$T(\mathbf{v}) = \mathbf{0}, \quad T(\mathbf{v}) = \mathbf{w}$$

and solve. Solutions $\mathbf{v}$ to the first equation are elements of $\operatorname{Ker} T$. Vectors $\mathbf{w}$, such that there exists a solution to the second equation, are elements of $\operatorname{Image} T$. In the process of solving, you will find yourself (maybe without realizing it) going through the process given above.

Theorem 6.2 (Rank-Nullity)**.** *If $T : V \to W$ is linear, and V is finite-dimensional, then*

$$\dim \operatorname{Ker} T + \dim \operatorname{Image} T = \dim V.$$

Corollary 6.3. *If $T : V \to W$ is linear, and V is finite-dimensional, then*

$$\dim \operatorname{Ker} T \geq \dim V - \dim W,$$
$$\dim \operatorname{Image} T \leq \dim V.$$

If T is one-to-one, then $\dim V \leq \dim W$*.*
If T is onto, then $\dim V \geq \dim W$*.*