Equivariant connections and quotient stacks I and II

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Topology Seminar
August 28, 2018

Goal

- For smooth manifolds equipped with Lie group action, “correctly” combine equivariance & connections on bundles and (higher) gerbes, and obtain secondary invariants.
- Demonstrate usefulness of ∞-sheaf language in geometric situations.

Note: Equivariant gerbes is joint with Byungdo Park (KIAS).

References

- R., “Differential Borel equivariant cohomology via connections”
- R., “An alternate description of equivariant connections”
- Byungdo Park and R. “A classification of equivariant gerbe connections”

Conventions/Notations

We work in Man, the category of smooth finite-dimensional manifolds

G, U - finite-dimensional Lie groups (later assumed to be compact)

some notation will assume $U = U(n)$

$M \in G$-Man - smooth manifold with G-action

$\mathrm{Bun}_{G,\nabla}(X)$ = groupoid of principal G-bundles with connection

$$= \begin{cases} \text{Objects:} & \begin{array}{ccc} G & \hookrightarrow & (P, \Theta) \\ & & \downarrow \\ & & X \end{array} \\ \text{Morphisms: bundle isomorphisms preserving connection} \end{cases}$$

G-$\mathrm{Bun}_{U,\nabla}(M)$ - G-equivariant U-bundles (Q, A) with G-invt connection;

i.e. $g^*A = A$ for every $g \in G$

ex. $(TM, LCconn)$ when G acts on M by isometries

$\mathrm{Bun}_U(X)$ = groupoid of principal U-bundles (without connection)

Chern classes

Characteristic classes are determined by the universal bundle.

Given $Q \in \mathrm{Bun}_U(M)$,

exists classifying map

$$M \xrightarrow{f} BU$$

$$\begin{aligned} H^*(M;\mathbb{Z}) &\xleftarrow{f^*} H^*(BU;\mathbb{Z}) \\ c_k(Q) &\longleftarrow\!\shortmid \quad c_k \end{aligned}$$

Chern–Weil

Given $(Q, A) \in \mathrm{Bun}_{U,\nabla}(M)$, obtain

$$\begin{tikzcd}
 & H^{2k}(M;\mathbb{Z}) \arrow[d] & \ni & c_k(Q) \arrow[ddll, mapsto, bend left] \\
\Omega^{2k}(M)_{\text{closed}} \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & H^{2k}(M;\mathbb{R}) & & \\
c_k(A) \arrow[r, mapsto] & \bullet & &
\end{tikzcd}$$

Chern–Simons

Suppose $M = M^3$ is a closed oriented 3-manifold.

Given $(Q, A) \in \mathrm{Bun}_{SU(n),\nabla}(M^3)$,

Choose section $s : M \to Q$, integrate Chern–Simons 3-form

$$\frac{1}{8\pi^2} \int_{M^3} s^* \operatorname{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \quad \in \quad \mathbb{R}/\mathbb{Z}$$

Where do these invariants really live? In $\widehat{H}^4(M)$.

Cheeger–Chern–Simons–Weil

Given $(Q, A) \in \mathrm{Bun}_{U,\nabla}(M)$, obtain

$$\begin{tikzcd}
\widehat{c}_k(Q,A) \arrow[r, phantom, "\in"] \arrow[rrr, mapsto, bend left] \arrow[dd, mapsto, bend right] & \widehat{H}^{2k}(M) \arrow[r] \arrow[d] & H^{2k}(M;\mathbb{Z}) \arrow[d] \arrow[r, phantom, "\ni"] & c_k(Q) \arrow[ddl, mapsto, bend left] \\
& \Omega^{2k}(M)_{\mathbb{Z}} \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & H^{2k}(M;\mathbb{R}) & \\
& c_k(A) \arrow[r, mapsto] & \bullet &
\end{tikzcd}$$

- $\Omega(M)_{\mathbb{Z}} =$ closed forms with $\mathbb{Z}$-periods; i.e.
  $$\Omega^n_{\mathbb{Z}}/d\Omega^{n-1} \cong \mathrm{Image}(H^n(\mathbb{Z}) \to H^n(\mathbb{R}))$$
- This is *not* a pullback in sets. It arises from a *homotopy pullback*.

Differential cohomology (Deligne cohomology)

Definition/Theorem (Cheeger–Simons, Deligne, ...)

Functors $\widehat{H}^* \colon \mathsf{Man}^{\mathrm{op}} \to \mathsf{GradedRings}$ satisfying:

- $$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{n-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B"] \arrow[dr] & & H^n(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
H^{n-1}(M;\mathbb{R}) \arrow[ur] \arrow[dr] & & \widehat{H}^n(M) \arrow[ur] \arrow[dr] & & H^n(M;\mathbb{R}) \\
& \dfrac{\Omega^{n-1}(M)}{\Omega^{n-1}(M)_{\mathbb{Z}}} \arrow[rr, "d"] \arrow[ur] & & \Omega^n(M)_{\mathbb{Z}} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

  where diagonals are short exact sequences,
- Chern–Weil homomorphism factors through $\widehat{H}^*$.

Chern–Simons (revisited)

Suppose $M = M^3$ is a closed oriented 3-manifold. Using short exact sequences,

$$H^4(M^3; \mathbb{Z}) = \Omega^4(M^3) = 0 \quad \Rightarrow \quad \widehat{H}^4(M^3) \cong \mathbb{R}/\mathbb{Z}.$$

Given $(Q, A) \in \mathrm{Bun}_{SU(n),\nabla}(M^3)$,

Choose section $s : M \to Q$, integrate Chern–Simons 3-form

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \cong & \widehat{H}^4(M^3) \\ \dfrac{1}{8\pi^2} \displaystyle\int_{M^3} s^* \operatorname{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \mod \mathbb{Z} & \leftrightarrow & \widehat{c}_2(Q, A) \end{array}$$

Differential cohomology (continued)

"Geometric" interpretation

$$
\begin{aligned}
\widehat{H}^1(M) &\cong C^\infty(M, S^1) \\
\widehat{H}^2(M) &\cong \mathrm{Bun}_{S^1,\nabla}(M)_{/\cong} \\
\widehat{H}^3(M) &\cong \mathrm{Grb}_\nabla(M)_{/\cong} \\
\widehat{H}^{p+2}(M) &\cong p\mathrm{Grb}_\nabla(M)_{/\cong}
\end{aligned}
$$

Remark

$\mathrm{Grb}_\nabla(M)$ is the 2-groupoid of S^1-banded gerbes with connection. Let $S^1 = \mathbb{R}/\mathbb{Z}$.

Degree 2

Under isomorphism $\widehat{H}^2(M) \cong \mathrm{Bun}_{S^1,\nabla}(M)_{/\cong}$

The diagram

$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^1(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B"] \arrow[dr] & & H^2(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
H^1(M;\mathbb{R}) \arrow[ur] \arrow[dr] & & \widehat{H}^2(M) \arrow[ur] \arrow[dr] & & H^2(M;\mathbb{R}) \\
& \dfrac{\Omega^1(M)}{\Omega^1(M)_{\mathbb{Z}}} \arrow[rr, "d"'] \arrow[ur] & & \Omega^2(M)_{\mathbb{Z}} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}
$$

Degree 2

Under isomorphism $\widehat{H}^2(M) \cong \mathrm{Bun}_{S^1,\nabla}(M)_{/\cong}$

... is isomorphic to

$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& \mathrm{Bun}_{S^1,\nabla}(M)_{\mathrm{flat}\ /\cong} \arrow[rr] \arrow[dr] & & \mathrm{Bun}_{S^1}(M)_{/\cong} \arrow[ur] \arrow[dr] & \\
H^1(M;\mathbb{R}) \arrow[ur] \arrow[dr] & & \mathrm{Bun}_{S^1,\nabla}(M)_{/\cong} \arrow[ur] \arrow[dr] & & H^2(M;\mathbb{R}) \\
& \dfrac{\Omega^1(M;\mathfrak{s}^1)}{\text{gauge trans.}} \arrow[ur] \arrow[rr] & & \{\text{curvature 2-forms}\} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}
$$

Differential cohomology highlights

- Encodes interesting secondary invariants.
- Describes isomorphism classes of higher geometric structures.
- “Compartmentalizes” aspects of these structures.

Desirable properties of G-equivariant U-connections

For G compact, equivariant connections always exist.

$$G\text{-Bun}_{U,\nabla}(M) \twoheadrightarrow G\text{-Bun}_U(M)$$

Equivariant Dirac-type operators, equivariant index theory, ...

Equivariant Chern–Weil (Borel, Berline–Vergne)

Given $(Q, A) \in G\text{-Bun}_{U,\nabla}(M)$, obtain

$$\begin{tikzcd}
 & H^{2k}_G(M;\mathbb{Z}) \arrow[d] & \ni & c_k(Q_G) \arrow[ddll, mapsto, bend left] \\
\Omega^{2k}_G(M)_{\text{closed}} \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & H^{2k}_G(M;\mathbb{R}) & & \\
c_k(A_G) \arrow[r, mapsto] & \bullet & &
\end{tikzcd}$$

Borel's equivariant cohomology

$M \in G$-Man
EG contractible with free G-action; i.e. $EG \to BG$ universal G-bundle

Borel: Replace M by $EG \times M$ and study its quotient.

$$H^n_G(M; -) := H^n(EG \times_G M; -)$$

Weil/Cartan de Rham models

$$\Omega_G(M) := \left(S\mathfrak{g}^* \otimes \Lambda\mathfrak{g}^* \otimes \Omega(M)\right)_{G-\text{basic}} \cong \left(S\mathfrak{g}^* \otimes \Omega(M)\right)^G$$
$$|S^1\mathfrak{g}^*| = 2, \quad |\Lambda^1\mathfrak{g}^*| = 1$$

Exists natural map, which is an isomorphism for compact G,

$$H^*(\Omega_G(M), d_G) \xrightarrow{\cong} H^*_G(M; \mathbb{R}).$$

Differential equivariant cohomology $\widehat{H}^*_G(M)$

Idea

To obtain differential equivariant cohomology, take the differential cohomology of the differential quotient stack.

Definition

$$\widehat{H}^n_G(M) := \widehat{H}^n(\mathcal{E}_\nabla G \times_G M)$$

What does this mean?

To be answered after showing *why* it is “a correct” definition.

Differential equivariant cohomology

Theorem (R., and Kübel independently)

For G compact, exist functors $\widehat{H}^*_G \colon G\text{-}\mathsf{Man}^{\mathrm{op}} \to \mathsf{GrRings}$ satisfying:

- $$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{n-1}_G(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B_G"] \arrow[dr] & & H^n_G(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
H^{n-1}_G(M;\mathbb{R}) \arrow[ur] \arrow[dr] & & \widehat{H}^n_G(M) \arrow[ur] \arrow[dr] & & H^n_G(M;\mathbb{R}) \\
& \dfrac{\Omega^{n-1}_G(M)}{\Omega^{n-1}_G(M)_{\mathbb{Z}}} \arrow[rr, "d_G"] \arrow[ur] & & \Omega^n_G(M)_{\mathbb{Z}} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

  where diagonals are short exact sequences,
- Equivariant Chern–Weil homomorphism factors through $\widehat{H}^*_G$.

(When $G = 1$, this recovers original differential cohomology.)

Differential equivariant cohomology

Theorem (R., Park–R.)

For G compact, $M \in G$-Man,

$$\begin{aligned} \widehat{H}^1_G(M) &\cong C^\infty(M, S^1)^G \\ \widehat{H}^2_G(M) &\cong G\text{-Bun}_{S^1,\nabla}(M)_{/\cong} \\ \widehat{H}^3_G(M) &\cong G\text{-Grb}_\nabla(M)_{/\cong} \end{aligned}$$

Sheaves/Stacks – things that naturally pullback

Definition

A *presheaf of infinity-groupoids on the site of manifolds* is a contravariant functor

$$\mathcal{F}\colon \mathsf{Man}^{\mathsf{op}} \to \mathsf{Gpd}_\infty$$

Sheaves $\mathsf{Shv}_\infty \subset \mathsf{Pre}_\infty$, which are a full subcategory, satisfy an additional sheaf/descent condition.

We use the natural inclusions

$$\mathsf{Set} \hookrightarrow \mathsf{Gpd} \hookrightarrow \mathsf{Gpd}_2 \hookrightarrow \mathsf{Gpd}_\infty \hookleftarrow \mathsf{Ch}_{\geq 0}.$$

Groupoid = category whose morphisms are all invertible
$\mathsf{Gpd}_\infty = \mathsf{Kan} \subset \mathsf{sSet}$

Cover

Surjective submersion $Y \xrightarrow{\pi} X$, which gives rise to simplicial manifold $\cdots Y \times_X Y \rightrightarrows Y$. For Y an open cover, this is Cech nerve.

$\mathcal{F} \in \mathrm{Pre}_\infty$ is a sheaf if for every cover $Y \to X$

$$\mathcal{F}(X) \xrightarrow{\simeq} \operatorname*{holim}_{\Delta} \left[\; \mathcal{F}(Y) \; \substack{\longrightarrow \\ \leftarrow\text{-}\text{-} \\ \longrightarrow} \; \mathcal{F}(Y^{[2]}) \; \substack{\longrightarrow \\ \leftarrow\text{-}\text{-} \\ \longrightarrow \\ \leftarrow\text{-}\text{-} \\ \longrightarrow} \; \mathcal{F}(Y^{[3]}) \;\; \cdots \right]$$

is a weak equivalence.

Sheaf condition is equivalent to

- usual sheaf condition when $\mathcal{F}$ is Set-valued
- stack condition when $\mathcal{F}$ is Gpd-valued

Sheafification $\mathbb{L} \colon \mathrm{Pre}_{\mathcal{C}} \to \mathrm{Shv}_{\mathcal{C}}$ exists, is left-adjoint to inclusion

$$\mathrm{Shv}_\infty(\mathbb{L}(\mathcal{F}), \mathcal{F}') \cong \mathrm{Pre}_\infty(\mathcal{F}, \mathcal{F}').$$

Examples

$\mathcal{F} \in \mathsf{Shv}_\infty$, X a test manifold, $\mathcal{F}(X) \in \mathsf{Gpd}_\infty$

- $M \in \mathsf{Man} \rightsquigarrow M \in \mathsf{Shv}_\infty$, by $M(X) := C^\infty(X, M) \in \mathsf{Set}$
- $\Omega^n \in \mathsf{Shv}_\infty$, by $\Omega^n(X) := \Omega^n(X) \in \mathsf{Set}$
- $\mathcal{B}U \in \mathsf{Shv}_\infty$, by $(\mathcal{B}U)(X) := \mathsf{Bun}_U(X) \in \mathsf{Gpd}$
- $\mathcal{B}_\nabla U \in \mathsf{Shv}_\infty$, by $(\mathcal{B}_\nabla U)(X) := \mathsf{Bun}_{U,\nabla}(X) \in \mathsf{Gpd}$
- $M \in G\text{-}\mathsf{Man} \rightsquigarrow$ *differential quotient stack* $\mathcal{E}_\nabla G \times_G M \in \mathsf{Shv}_\infty$

$$(\mathcal{E}_\nabla G \times_G M)(X) = \begin{cases} \text{Obj:} \quad \begin{array}{ccc} (P,\Theta) & \xrightarrow{f} & M \\ \downarrow & & \\ X & & \end{array} \qquad \begin{array}{c} (P,\Theta) \in \mathsf{Bun}_{G,\nabla}(X) \\ \\ f \text{ is } G\text{-equivariant} \end{array} \\ \text{Mor: preserve } \Theta, \text{ compatible with } f \end{cases}$$

If $M = \mathrm{pt}$, then $\mathcal{E}_\nabla G \times_G \mathrm{pt} = \mathcal{B}_\nabla G$.

- $\mathcal{E}G \times_G M \in \mathsf{Shv}_\infty$ defined similarly, but without connections

Maps between sheaves

Yoneda Lemma: Manifolds embed into Sheaves

$$\mathrm{Shv}_\infty(M, \mathcal{F}) \cong \mathcal{F}(M) \in \mathrm{Gpd}_\infty$$

Examples

- Sheaf maps generalize smooth maps between manifolds

$$\begin{tikzcd}
 & \mathrm{Man}(M,N) \arrow[r, phantom, "="] & C^\infty(M,N) \arrow[d, phantom, "\parallel"] \\
\mathrm{Map}(M,N) \arrow[ur, phantom, "\overset{?}{\cong}"] \arrow[dr, phantom, "\overset{?}{\cong}"] & & \\
 & \mathrm{Shv}_\infty(M,N) \arrow[r, phantom, "\cong"] & N(M)
\end{tikzcd}$$

- $\omega \in \Omega^n(M)$ is equivalent to $M \xrightarrow{\omega} \Omega^n$
- $(Q, A) \in \mathrm{Bun}_{U,\nabla}(M)$ is equivalent to $M \xrightarrow{(Q,A)} \mathcal{B}_\nabla U$.
  Canonical classifying maps!

Differential Forms

Since $\Omega^n(M) \cong \mathsf{Shv}_\infty(M, \Omega^n)$, for $\mathcal{F} \in \mathsf{Shv}_\infty$ *define*

$$\Omega^n(\mathcal{F}) := \mathsf{Shv}_\infty(\mathcal{F}, \Omega^n) \in \mathsf{Set}.$$

Unpackaging, what is a differential form $\omega \in \Omega^n(\mathcal{B}_\nabla G)$?

$$\begin{array}{ccc} (\mathcal{B}_\nabla G)(X) & \xrightarrow{\omega} & \Omega^n(X) \\ \downarrow f^* & & \downarrow f^* \\ (\mathcal{B}_\nabla G)(Y) & \xrightarrow{\omega} & \Omega^n(Y) \end{array}$$

In other words,

- to any $(P, \Theta) \to X$, we get $\omega(P, \Theta) \in \Omega^n(X) \quad \in \mathsf{Set}$,
- if $(P, \Theta) \cong (P', \Theta')$, then $\omega(P, \Theta) = \omega(P', \Theta')$,
- $\omega(f^*(P, \Theta)) = f^*(\omega(P, \Theta))$.

Differential Forms (continued)

$\Omega^n(\mathcal{B}_\nabla G) := \mathrm{Shv}_\infty(\mathcal{B}_\nabla G, \Omega^n)$

Observation

Chern–Weil construction gives homomorphism

$$\begin{aligned}(S^k\mathfrak{g}^*)^G &\xrightarrow{\cong} \Omega^{2k}(\mathcal{B}_\nabla G)\\ \omega &\longmapsto \Big((P,\Theta) \mapsto \omega(\Omega^{\wedge k})\Big)\end{aligned}$$

Here, $(P,\Theta) \in \mathrm{Bun}_{G,\nabla}(X)$ with curvature $\Omega \in \Omega^2(P;\mathfrak{g})$, and $\omega(\Omega^{\wedge k}) \in \pi^*\Omega^{2k}(X)$.

Theorem (Freed–Hopkins)

This map is an *isomorphism* of sets, abelian groups. (bijection)

“Goldilocks phenomenon”

- $\Omega(BG)$ - too big
- $\Omega(\mathcal{B}G)$ - too small, $\Omega^n(\mathcal{B}G) = 0$ for $n > 0$
- $\Omega(\mathcal{B}_\nabla G) \cong (S\mathfrak{g}^*)^G$ - just right

Equivariant forms

$$\Omega^*_G(M) := \big(S\mathfrak{g}^* \otimes \Lambda\mathfrak{g}^* \otimes \Omega(M)\big)_{G\text{-basic}}, \quad |S^1\mathfrak{g}^*| = 2,\ |\Lambda^1\mathfrak{g}^*| = 1.$$

Given $(P, \Theta, f) \in (\mathcal{E}_\nabla G \times_G M)(X)$, Weil homomorphism gives

$$\begin{array}{ccc} (P,\Theta) & \xrightarrow{f} & M \\ \downarrow & & \\ X & & \end{array} \qquad \begin{array}{ccc} \Omega_G(M) \xrightarrow{f^*} \Omega_G(P) & \xrightarrow{\Theta^*} & \Omega(P)_{\text{bas}} \cong \Omega(X) \\ \alpha\beta\gamma & \longmapsto & \alpha(\Omega^{\wedge i}) \wedge \beta(\Theta^{\wedge j}) \wedge f^*\gamma \end{array}$$

Rewritten,

$$X \xrightarrow{(P,\Theta,f)} \mathcal{E}_\nabla G \times_G M \quad \Rightarrow \quad \Omega_G(M) \xrightarrow{(P,\Theta,f)^*} \Omega(X)$$

Theorem (Freed–Hopkins)

$$\Omega_G(M) \xrightarrow{\cong} \Omega(\mathcal{E}_\nabla G \times_G M)$$

Cohomology as a sheaf

Cohomology

For discrete abelian group A, define $\mathcal{K}(A,n) \in \mathsf{Shv}_\infty$ so that

$$H^n(M;A) \cong \pi_0 \mathsf{Shv}_\infty(M, \mathcal{K}(A,n)) \cong \pi_0 \,\mathsf{Top}(M, K(A,n)).$$

Proposition

$$\begin{aligned}\mathsf{Shv}_\infty(\mathcal{E}_\nabla G \times_G M, \mathcal{K}(A,n)) &\simeq \mathsf{Shv}_\infty(\mathcal{E}G \times_G M, \mathcal{K}(A,n)) \\ &\simeq \mathsf{Top}(EG \times_G M, K(A,n))\end{aligned}$$

Hence,

$$H^n(\mathcal{E}_\nabla G \times_G M; A) := \pi_0 \mathsf{Shv}_\infty(\mathcal{E}_\nabla G \times_G M, \mathcal{K}(A,n)) \cong H^n_G(M;A)$$

Differential cohomology as a sheaf

Form homotopy pullback square in Shv_∞

$$\begin{tikzcd} \widehat{\mathcal{K}}(\mathbb{Z}, n) \arrow[r] \arrow[d] & \mathcal{K}(\mathbb{Z}, n) \arrow[d] \\ \Omega^n_{\text{closed}} \arrow[r] & \mathcal{K}(\mathbb{R}, n) \end{tikzcd}$$

Differential cohomology is represented by maps to $\widehat{\mathcal{K}}(\mathbb{Z}, n)$.

Definition

For $\mathcal{F} \in \mathsf{Shv}_\infty$, define $\widehat{H}^n(\mathcal{F}) := \pi_0 \mathsf{Shv}_\infty(\mathcal{F}, \widehat{\mathcal{K}}(\mathbb{Z}, n))$. Then,

$$\widehat{H}^n_G(M) := \widehat{H}^n(\mathcal{E}_\nabla G \times_G M) := \pi_0 \mathsf{Shv}_\infty \left(\mathcal{E}_\nabla G \times_G M, \widehat{\mathcal{K}}(\mathbb{Z}, n) \right)$$

Differential Equivariant Cohomology

Abbreviate $\mathcal{E}_\nabla G \times_G M$ by $M /\!\!/_\nabla G$,

Theorem (R.): Let $M \in G$-Man, G a compact Lie group

Our diagram ...

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{n-1}_G(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B_G"] \arrow[dr] & & H^n_G(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
H^{n-1}_G(M;\mathbb{R}) \arrow[ur] \arrow[dr] & & \widehat{H}^n_G(M) \arrow[ur] \arrow[dr] & & H^n_G(M;\mathbb{R}) \\
& \dfrac{\Omega^{n-1}_G(M)}{\Omega^{n-1}_G(M)_\mathbb{Z}} \arrow[rr, "d_G"] \arrow[ur] & & \Omega^n_G(M)_\mathbb{Z} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

Differential Equivariant Cohomology

Abbreviate $\mathcal{E}_\nabla G \times_G M$ by $M /\!\!/_\nabla G$,

Theorem (R.): Let $M \in G$-Man, G a compact Lie group

…is obtained from

$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
 & H^{n-1}(M /\!\!/_\nabla G; \mathbb{R}/\mathbb{Z}) \arrow[rr, "-B"] \arrow[dr] & & H^n(M /\!\!/_\nabla G; \mathbb{Z}) \arrow[ur] \arrow[dr] & \\
H^{n-1}(M /\!\!/_\nabla G; \mathbb{R}) \arrow[ur] \arrow[dr] & & \widehat{H}^n(M /\!\!/_\nabla G) \arrow[ur] \arrow[dr] & & H^n(M /\!\!/_\nabla G; \mathbb{R}) \\
 & \dfrac{\Omega^{n-1}(M /\!\!/_\nabla G)}{\Omega^{n-1}(M /\!\!/_\nabla G)_{\mathbb{Z}}} \arrow[rr, "d"] \arrow[ur] & & \Omega^n(M /\!\!/_\nabla G)_{\mathbb{Z}} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}
$$

Cheeger–Simons–Chern–Weil revisited

Characteristic classes have a natural differential refinement

$$0 \to \frac{\Omega^{2k-1}(\mathcal{B}_\nabla U)}{\Omega^{2k-1}(\mathcal{B}_\nabla U)_{\mathbb{Z}}} \longrightarrow \widehat{H}^{2k}(\mathcal{B}_\nabla U) \longrightarrow H^{2k}(\mathcal{B}_\nabla U;\mathbb{Z}) \to 0$$

$$0 \to 0 \longrightarrow \widehat{H}^{2k}(\mathcal{B}_\nabla U) \xrightarrow{\cong} H^{2k}(BU;\mathbb{Z}) \to 0$$

Given $(Q, A) \in \mathrm{Bun}_{U,\nabla}(X)$,

$$X \xrightarrow{(Q,A)} \mathcal{B}_\nabla U$$

$$\begin{array}{ccccc} \widehat{H}^{2k}(X) & \xleftarrow{(P,\Theta)^*} & \widehat{H}^{2k}(\mathcal{B}_\nabla U) & \cong & H^{2k}(BU;\mathbb{Z}) \\ \widehat{c}_k(Q,A) & \longleftarrow & \widehat{c}_k & \longmapsto & c_k \end{array}$$

Reduce to universal case and use “classifying maps”!

Equivariant connections and Chern–Weil theory (Weil model)

$(Q, A) \in G\text{-Bun}_{U,\nabla}(M)$ with ρ the induced $\mathfrak{g}$-action on TQ.

Define $\iota_{\rho(\theta_{\mathfrak{g}})}A \in \Lambda^1\mathfrak{g}^* \otimes \Omega^0(Q) \otimes \mathfrak{u}$ by $(\iota_{\rho(\theta_{\mathfrak{g}})}A)(\xi) = \iota_{\rho(\xi)}A$. (Here, $\theta_{\mathfrak{g}}$ is Maurer–Cartan 1-form.)

$$A_G := A - \iota_{\rho(\theta_{\mathfrak{g}})}A \quad \in \Omega^1_G(Q; \mathfrak{u})$$

$$F_G := d_G A_G + \tfrac{1}{2}[A_G, A_G] \quad \in \Omega^2_G(Q; \mathfrak{u}).$$

Definition (Berline–Vergne)

$$c_k^G(F) := c_k(F_G) \in \Omega^{2k}_G(M).$$

Equivariant connections and Chern–Weil theory (cont'd)

How to interpret $(Q, A) \in G\text{-Bun}_{U,\nabla}(M)$ in sheaf language?

Two simple but important observations.

- The first is a problem with using $G \times M \rightrightarrows M$.
- The second is the *solution*.

These illustrate the importance of using $\mathcal{E}_\nabla G \times_G M$, not $\mathcal{E}G \times_G M$.

Observation 1:

Action Lie groupoid $M /\!\!/ G := [G \times M \rightrightarrows M]$

Well-known equivalence

$$G\text{-Bun}_U(M) \xrightarrow{\cong} \text{Bun}_U(M /\!\!/ G)$$

$$\begin{tikzcd} Q \arrow[d] \\ M \end{tikzcd} \qquad \qquad \begin{tikzcd} G \times Q \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\ell"'] \arrow[d] & Q \arrow[d] \\ G \times M \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\ell"'] & M \end{tikzcd}$$

Remark: $M /\!\!/ G \overset{\text{Yoneda}}{\rightsquigarrow} M /\!\!/ G \in \text{Pre}_\infty$. Then $\mathbb{L}(M /\!\!/ G) \simeq \mathcal{E}G \times_G M \in \text{Shv}_\infty$. Hence,

$$\text{Pre}_\infty(M /\!\!/ G, \mathcal{B}U) \cong \text{Pre}_\infty(\mathcal{E}G \times_G M, \mathcal{B}U) = \text{Shv}_\infty(\mathcal{E}G \times_G M, \mathcal{B}U).$$

Observation 1: Problem with usual quotient stack

Compatible with G-invt connection A? In general, NO!

$$G\text{-Bun}_{U,\nabla}(M) \overset{?}{\dashrightarrow} \text{Bun}_{U,\nabla}(M /\!\!/ G)$$

$$\begin{tikzcd} (Q,A) \arrow[d] \\ M \end{tikzcd} \qquad \begin{tikzcd} G \times (Q,A) \arrow[r, shift left, "\pi"] \arrow[r, shift right, dotted, "\ell"'] \arrow[d] & (Q,A) \arrow[d] \\ G \times M \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\ell"'] & M \end{tikzcd}$$

Lemma: $G \times (Q,A) \xrightarrow{\ell} (Q,A)$ conn preserving iff A is G-basic

G-action on Q induces vector fields $\rho\colon \mathfrak{g} \to \Gamma(TQ)$. For $\xi \in \mathfrak{g} \cong T_eG$,

$$\iota_\xi(\ell^* A)_{(e,q)} = A_{(q)}(\rho(\xi)) = \iota_{\rho(\xi)} A_{(q)},$$

which equals 0 precisely when $A \in \Omega^1(Q,\mathfrak{u})^G_{\mathfrak{g}-\text{hor}} = \Omega^1(Q,\mathfrak{u})_{G\text{-bas}}$.

Observation 2: Key to Solution

Suppose $P \in G$-Man with $(P, \Theta) \in \mathrm{Bun}_{G,\nabla}(X)$.

Canonical construction of G-basic U-connections

$$
\begin{tikzcd}
G\text{-}\mathrm{Bun}_{U,\nabla}(P) \arrow[r] \arrow[rr, bend left, "\Theta^*"] & G\text{-}\mathrm{Bun}_{U,\nabla}(P)_{G\text{-bas}} \arrow[r] & \mathrm{Bun}_{U,\nabla}(P/G) \\
Q \arrow[rr, mapsto] & & Q/G \\
A \arrow[r, mapsto] & A - \iota_{\rho(\Theta)}A & \\
\Omega^1(Q;\mathfrak{u}) \arrow[u, phantom, "\Cap"] & \Omega^1(Q;\mathfrak{u})_{G\text{-bas}} \arrow[u, phantom, "\Cap"] \arrow[r, phantom, "\cong"] & \Omega^1(Q/G;\mathfrak{u})
\end{tikzcd}
$$

$Q \xrightarrow{\pi} P$, $\pi^*\Theta \in \Omega^1(Q, \mathfrak{g})$. If $v \in TQ$, then $(\pi^*\Theta)(v) \in \mathfrak{g}$, and

$$(\iota_{\rho(\Theta)}A)(v) := \iota_{\rho(\pi^*\Theta(v))}A = A\big(\rho(\Theta(\pi_* v))\big).$$

G-equivariant U-connections Revisited

Define $\mathrm{Bun}_{U,\nabla}(\mathcal{E}_\nabla G \times_G M) := \mathrm{Shv}_\infty(\mathcal{E}_\nabla G \times_G M, \mathcal{B}_\nabla U)$.

Given $X \xrightarrow{(P,\Theta,f)} \mathcal{E}_\nabla G \times_G M$, is there induced $X \to \mathcal{B}_\nabla U$?

$$G\text{-}\mathrm{Bun}_{U,\nabla}(M) \xrightarrow{f^*} G\text{-}\mathrm{Bun}_{U,\nabla}(P) \xrightarrow{\Theta^*} \mathrm{Bun}_{U,\nabla}(X)$$

$$(Q, A) \longmapsto \big(f^*Q/G, (1 - \iota_{\rho(\Theta)})f^*A\big)$$

Theorem (R.)

For $M \in G$-Man, this induces an equivalences of categories

$$G\text{-}\mathrm{Bun}_U(M) \cong \mathrm{Bun}_U(\mathcal{E}G \times_G M)$$
$$G\text{-}\mathrm{Bun}_{U,\nabla}(M) \cong \mathrm{Bun}_{U,\nabla}(\mathcal{E}_\nabla G \times_G M)$$
$$G\text{-}\mathrm{Bun}_{U,\nabla}(M)_{G\text{-bas}} \cong \mathrm{Bun}_{U,\nabla}(\mathcal{E}G \times_G M)$$

$$
\begin{tikzcd}
M \arrow[rr, "{(Q,\Theta)}"] \arrow[d] & & \mathcal{B}_{\nabla}K \arrow[d] \\
M/\!\!/_{\nabla}G \arrow[r] \arrow[urr, dotted, "\Theta\ G\text{-invt}"] & M/\!\!/G \arrow[r, dotted, "G\text{-struct on } Q"'] \arrow[ur, dotted, "\Theta\ G\text{-bas}"'] & \mathcal{B}K
\end{tikzcd}
$$

Equivariant Cheeger–Simons–Chern–Weil

Given $(Q, A) \in G\text{-Bun}_{U,\nabla}(M)$, with G, U compact,

$$\mathcal{E}_\nabla G \times_G M \xrightarrow{(Q,A)_G} \mathcal{B}_\nabla U$$

$$\begin{array}{ccccc} \widehat{H}^{2n}_G(M) = \widehat{H}^{2n}(\mathcal{E}_\nabla G \times_G M) & \xleftarrow{(Q,A)^*_G} & \widehat{H}^{2n}(\mathcal{B}_\nabla U) & \cong & H^{2n}(BU;\mathbb{Z}) \\ \widehat{c}_n((Q,A)_G) & \longleftarrow\!\shortmid & \widehat{c}_n & \longleftarrow\!\shortmid & c_n \end{array}$$

Theorem (R., Freed)

$\widehat{c}_n((Q, A)_G)$ maps to the traditional equivariant characteristic class and equivariant form under natural homomorphisms $\widehat{H}^{2n}_G(M) \to H^{2n}_G(M;\mathbb{Z})$ and $\widehat{H}^{2n}_G(M) \to \Omega^{2n}_G(M)$.

Equivariant Chern–Weil forms (Weil model)

$(Q, A) \in G\text{-Bun}_{U,\nabla}(M)$ with ρ the induced $\mathfrak{g}$-action on Q.

Define $\iota_{\rho(\theta_{\mathfrak{g}})}A \in \Lambda^1\mathfrak{g}^* \otimes \Omega^0(Q) \otimes \mathfrak{u}$ by $(\iota_{\rho(\theta_{\mathfrak{g}})}A)(\xi) = \iota_{\rho(\xi)}A$. (Here, $\theta_{\mathfrak{g}}$ is Maurer–Cartan 1-form.)

$$\begin{aligned} A_G &:= A - \iota_{\rho(\theta_{\mathfrak{g}})}A \quad &\in \Omega^1_G(Q; \mathfrak{u}) \\ F_G &:= d_G A_G + \tfrac{1}{2}[A_G, A_G] \quad &\in \Omega^2_G(Q; \mathfrak{u}). \end{aligned}$$

Definition (Berline–Vergne)

$$c_k^G(F) := c_k(F_G) \in \Omega^{2k}_G(M).$$

This $(1 - \iota_{\rho(\theta_{\mathfrak{g}})})A$ becomes the $(1 - \iota_{\rho(\Theta)})f^*A$ under the isomorphism $\Omega_G(Q) \cong \Omega(\mathcal{E}_\nabla G \times_G Q)$.

Specific case $U = S^1$

Natural equivalence $\mathcal{B}_\nabla S^1 \simeq \widehat{\mathcal{K}}(\mathbb{Z}, 2)$. This is most easily seen using the Deligne complex to model differential cohomology. Using Dold–Kan Γ and sheafification $\mathbb{L}$,

$$\mathcal{B}_\nabla S^1 \simeq \mathbb{L}(\Gamma(S^1 \xrightarrow{d\log} \Omega^1)) \simeq \widehat{\mathcal{K}}(\mathbb{Z}, 2) \in \mathsf{Shv}_\infty.$$

Reminder - sheaf condition

$\mathcal{F} \in \mathsf{Pre}_\infty$ is a sheaf if for every cover $Y \to X$

$$\mathcal{F}(X) \xrightarrow{\simeq} \operatorname*{holim}_{\Delta} \left[\mathcal{F}(Y) \rightrightarrows \mathcal{F}(Y^{[2]}) \,\substack{\longrightarrow\\[-2pt] \longrightarrow\\[-2pt] \longrightarrow}\, \mathcal{F}(Y^{[3]}) \ \cdots \right]$$

is a weak equivalence.

For $\mathcal{U} = \coprod U_i \twoheadrightarrow M$ open cover, and $\mathcal{F} = (S^1 \xrightarrow{d\log} \Omega^1)$, then ...

Specific case $U = S^1 = \mathbb{R}/\mathbb{Z}$ continued

For $\mathcal{U} = \coprod U_i \twoheadrightarrow M$ open cover, and $\mathcal{F} = (S^1 \xrightarrow{d\log} \Omega^1)$, then

$$\operatorname*{holim}_{\Delta} \left[\mathcal{F}(\textstyle\coprod U_i) \rightrightarrows \mathcal{F}(\textstyle\coprod U_i \cap U_j) \substack{\longrightarrow\\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \mathcal{F}(U_i \cap U_j \cap U_k) \quad \cdots \right]$$

is given by the total complex of the double-complex

$$\begin{array}{ccccc}
\Omega^1(U_i) & \xrightarrow{\ \delta\ } & \Omega^1(U_{ij}) & \xrightarrow{\ \delta\ } & \Omega^1(U_{ijk}) \cdots \\
d\uparrow & & d\uparrow & & d\uparrow \\
C^\infty(U_i, S^1) & \stackrel{\delta}{\to} & C^\infty(U_{ij}, S^1) & \stackrel{\delta}{\to} & C^\infty(U_{ijk}, S^1) \cdots
\end{array}$$

Object is: (g_{ij}, A_i), where $g_{ij} \in C^\infty(U_{ij}, S^1)$ such that $g_{ij} g_{jk} g_{ki} = 1$, and $A \in \Omega^1(U_i)$ such that $A_i - A_j = dg_{ij}$.

Under isomorphism $\widehat{H}^2_G(M) \cong G\text{-}\mathrm{Bun}_{S^1,\nabla}(M)_{/\cong}$

The diagram

$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^1_G(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B_G"] \arrow[dr] & & H^2_G(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
H^1_G(M;\mathbb{R}) \arrow[ur] \arrow[dr] & & \widehat{H}^2_G(M) \arrow[ur] \arrow[dr] & & H^2_G(M;\mathbb{R}) \\
& \dfrac{\Omega^1_G(M)}{\Omega^1_G(M)_{\mathbb{Z}}} \arrow[ur] \arrow[rr, "d_G"'] & & \Omega^2_G(M)_{\mathbb{Z}} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}
$$

Under isomorphism $\widehat{H}^2_G(M) \cong G\text{-Bun}_{S^1,\nabla}(M)_{/\cong}$

... is isomorphic to

$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& G\text{-Bun}_{S^1,\nabla}(M)_{\text{flat}\ /\cong} \arrow[rr] \arrow[dr] & & G\text{-Bun}_{S^1}(M)_{/\cong} \arrow[ur] \arrow[dr] & \\
H^1_G(M;\mathbb{R}) \arrow[ur] \arrow[dr] & & G\text{-Bun}_{S^1,\nabla}(M)_{/\cong} \arrow[ur] \arrow[dr] & & H^2_G(M;\mathbb{R}) \\
& \dfrac{\Omega^1_G(M;\mathfrak{s}^1)}{\text{gauge trans.}} \arrow[rr] \arrow[ur] & & \{\text{equiv curvs}\} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}
$$

Background: Abelian gerbes

S^1-banded gerbes with connection on M

admit multiple explicit models, which are all equivalent to the 2-groupoid of cocycles in the Deligne complex

$$\mathbb{L}(\Gamma(S^1 \xrightarrow{d\log} \Omega^1 \xrightarrow{d} \Omega^2)) \simeq \mathcal{B}^2_\nabla S^1 \simeq \widehat{\mathcal{K}}(\mathbb{Z},3) \in \mathrm{Shv}_\infty.$$

Using any of these gives a definition or theorem

$$\mathrm{Grb}_\nabla(M) \cong \mathrm{Shv}_\infty(M, \mathcal{B}^2_\nabla S^1).$$

More generally,

$$p\mathrm{Grb}_\nabla(M) \cong \mathrm{Shv}_\infty(M, \mathcal{B}^{p+1}_\nabla S^1).$$

Equivariant gerbe connections

If one defines equivariant gerbe connections on M as the gerbe connections on the differential quotient stack

$$\mathrm{Grb}_\nabla(\mathcal{E}_\nabla G \times_G M) := \mathrm{Shv}_\infty(\mathcal{E}_\nabla G \times_G M, \mathcal{B}^2_\nabla S^1),$$

then we get (for free) the desirable classification

$$\pi_0 \mathrm{Grb}_\nabla(\mathcal{E}_\nabla G \times_G M) \cong \widehat{H}^3_G(M),$$

with nice short exact sequences, justifying this as a “correct definition.”

Question

Is there a more concrete geometric model?

Equivariant bundle gerbe connections

For $G = 1$, this is the definition of a "bundle gerbe with connection and curving" on M.

Definition (Meinrenken): Let $M \in G$-Man, with G compact.

A G-equivariant bundle gerbe with equivariant connection consists of:

- G-equivariant surjective submersion $\pi\colon Y \to M$
- $(Q, A) \in G\text{-Bun}_{S^1,\nabla}(Y^{[2]})$
- $B \in \Omega^2_G(Y)$ satisfying $\mathrm{curv}_G(A) = \delta B \in \Omega^2_G(Y^{[2]})$
- isomorphism $\mu : \pi^*_{12}(Q, A) \otimes \pi^*_{23}(Q, A) \to \pi^*_{13}(Q, A)$ in $G\text{-Bun}_{S^1,\nabla}(Y^{[3]})$, associative over $Y^{[4]}$.

Concisely:

$$\widehat{\mathcal{L}} = (Y, Q, A, B, \mu) \in G\text{-Grb}_\nabla(M)$$

$$S^1 \to (Q, A) \to Y^{[2]} \rightrightarrows (Y, B) \xrightarrow{\pi} M$$

Equivariant bundle gerbe connections (cont'd)

Definition (Waldorf, Park–R.):

An isomorphism $\widehat{\mathcal{K}} = (Z, K, \nabla, \alpha) \colon \widehat{\mathcal{L}}_1 \to \widehat{\mathcal{L}}_2 \in G\text{-Grb}_\nabla(M)$ is:

- G-equiv surj subm $\zeta \colon Z \to Y_1 \times_M Y_2$,
- $(K, \nabla) \in G\text{-Bun}_{S^1,\nabla}(Z)$, with $\mathrm{curv}_G(\nabla) = \zeta^*(B_2 - B_1) \in \Omega^2_G(Z)$,
- iso $\alpha \colon \delta(K, \nabla) \to (Q_1, A_1)^{-1} \otimes (Q_2, A_2)$ compatible with μ_1, μ_2.

Remarks:

- Also have 2-morphisms, making $G\text{-Grb}_\nabla(M)$ a 2-groupoid.
- For $G = 1$ this is Waldorf's 2-groupoid $\mathrm{Grb}_\nabla(M)$, and isomorphisms are equivalent to the stable isomorphisms of Murray–Stevenson.

Relating the two models

Want: $G\text{-}\mathrm{Grb}_\nabla(M) \to \mathrm{Grb}_\nabla(\mathcal{E}_\nabla G \times_G M)$

For $(P,\Theta) \in \mathrm{Bun}_{G,\nabla}(X)$, the previous $G\text{-}\mathrm{Bun}_{S^1,\nabla}(P) \xrightarrow{\Theta^*} \mathrm{Bun}_{S^1,\nabla}(X)$ induces

$$G\text{-}\mathrm{Grb}_\nabla(P) \xrightarrow{\Theta^*} \mathrm{Grb}_\nabla(X).$$

Given $\widehat{\mathcal{L}} \in G\text{-}\mathrm{Grb}_\nabla(M)$, define $\widehat{\mathscr{L}} \in \mathrm{Shv}_\infty(\mathcal{E}_\nabla G \times_G M, \mathcal{B}^2_\nabla S^1)$ by

$$\begin{aligned}
(\mathcal{E}_\nabla G \times_G M)(X) &\longrightarrow (\mathcal{B}^2_\nabla S^1)(X) \\
(P,\Theta,f) &\longmapsto \Theta^*(f^*(\widehat{\mathcal{L}}))
\end{aligned}$$

$$\begin{array}{ccc}
(P,\Theta) & \xrightarrow{f} & M \\
\downarrow & & \\
X & &
\end{array}
\qquad
\begin{array}{ccccc}
G\text{-}\mathrm{Grb}_\nabla(M) & \xrightarrow{f^*} & G\text{-}\mathrm{Grb}_\nabla(P) & \xrightarrow{\Theta^*} & \mathrm{Grb}_\nabla(X) \\
\widehat{\mathcal{L}} & & \longmapsto & & \widehat{\mathscr{L}}(P,\Theta,f)
\end{array}$$

Theorem (Park–R.)

Given $M \in G$-Man with G compact, the natural map

$$G\text{-}\mathrm{Grb}_\nabla(M) \xrightarrow{\simeq} \mathrm{Grb}_\nabla(\mathcal{E}_\nabla G \times_G M)$$

is an equivalence of 2-categories, with isomorphisms …

$$\begin{tikzcd}
0 \arrow[drr] & & & & & & & & 0 \\
& & H^2_G(M;\mathbb{R}/\mathbb{Z}) \arrow[rrrr, "-B"] \arrow[drr] & & & & H^3_G(M;\mathbb{Z}) \arrow[urr] \arrow[drr] & & \\
H^2_G(M;\mathbb{R}) \arrow[urr] \arrow[drr] & & & & \widehat{H}^3_G(M) \arrow[urr] \arrow[drr] & & & & H^3_G(M;\mathbb{R}) \\
& & \dfrac{\Omega^2_G(M)}{\Omega^2_G(M)_{\mathbb{Z}}} \arrow[urr] \arrow[rrrr, "d_G"'] & & & & \Omega^3_G(M)_{\mathbb{Z}} \arrow[urr] \arrow[drr] & & \\
0 \arrow[urr] & & & & & & & & 0
\end{tikzcd}$$

Theorem (Park–R.)

Given $M \in G$-Man with G compact, the natural map

$$G\text{-Grb}_\nabla(M) \xrightarrow{\simeq} \text{Grb}_\nabla(\mathcal{E}_\nabla G \times_G M)$$

is an equivalence of 2-categories, with isomorphisms . . .

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& G\text{-Grb}_\nabla(M)_{\text{flat}}{}_{/\cong} \arrow[rr] \arrow[dr] & & G\text{-Grb}(M)_{/\cong} \arrow[ur] \arrow[dr] & \\
H^2_G(M;\mathbb{R}) \arrow[ur] \arrow[dr] & & G\text{-Grb}_\nabla(M)_{/\cong} \arrow[ur, "\mathrm{DD}_G"] \arrow[dr, "\mathrm{curv}_G"'] & & H^3_G(M;\mathbb{R}) \\
& G\text{-Grbtriv}_\nabla(M)_{/\cong} \arrow[ur] \arrow[rr] & & \Omega^3_G(M)_{\mathbb{Z}} \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

Upshot: Easy existence/uniqueness results

Gerbes on G

- G be compact semisimple Lie group
- $\eta \in H^3(G;\mathbb{Z})$
- $\omega = \langle \theta, [\theta,\theta]\rangle_\eta \in \Omega^3(G)$ de Rham representative of η

Proposition

Then, (η, ω_η) determines a gerbe connection on G, unique up to isomorphism.

Proof

$H^2(G;\mathbb{R}) = 0$ and the short exact sequence

$$0 \to \frac{H^2(G;\mathbb{R})}{H^2(G;\mathbb{R})_{\mathbb{Z}}} \to \widehat{H}^3(G) \to H^3(G;\mathbb{Z}) \times_{\bullet} \Omega^3(G)_{\mathbb{Z}} \to 0.$$

Let G act on itself by conjugation. $H^3_G(G;\mathbb{R}) \cong H^3(G;\mathbb{R})$, and the form ω has a natural equivariant extension $\omega_\eta \in \Omega^3_G(G)$.

Theorem (Park–R.)

Suppose η has an equivariant extension $\eta_G \in H^3_G(G;\mathbb{Z})$. Then there exists an equivariant gerbe connection $\widehat{\mathcal{L}}_\eta \in G\text{-}\mathrm{Grb}_\nabla(G)$ satisfying

$$\mathrm{DD}_G(\widehat{\mathcal{L}}_\eta) = \eta_G \in H^3_G(G;\mathbb{Z}), \quad \mathrm{curv}_G(\widehat{\mathcal{L}}_\eta) = \omega \in \Omega^3_G(G),$$

and such an $\widehat{\mathcal{L}}_\eta$ is unique up to isomorphism.

Proof

$H^2_G(G;\mathbb{R}) = 0$, and the short exact sequence

$$0 \to \frac{H^2_G(G;\mathbb{R})}{H^2_G(G;\mathbb{R})_\mathbb{Z}} \to \widehat{H}^3_G(G) \to H^3_G(G;\mathbb{Z}) \times_\bullet \Omega^3_G(G)_\mathbb{Z} \to 0.$$