Differential Equivariant Cohomology

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Union College Mathematics Conference
December 3, 2016

Goal

- Obtain secondary invariants for G-equivariant vector bundles with connection
- by refining equivariant Chern–Weil theory
- in a natural way.

References

- R., "Differential Borel equivariant cohomology via connections" [arXiv:1602.06921]
- R., "An alternate description of equivariant connections" [arXiv:1608.01297]

Chern–Weil

Given $(E, \nabla) \in \mathrm{Bun}_{U,\nabla}(M)$

$$\begin{array}{ccc} (E, \nabla) & & \text{Hermitian vector bundle with connection} \\ \downarrow & & \\ M & & \text{smooth manifold} \end{array}$$

Obtain

$$\begin{array}{ccccc} & & H^{2k}(M;\mathbb{Z}) & \ni & c_k(E) \\ & & \downarrow & & \\ \Omega^{2k}(M)_{\text{closed}} & \longrightarrow & H^{2k}(M;\mathbb{R}) & & \\ \Cup & & & & \\ c_k(\nabla) & \longmapsto & \bullet & \longleftarrow\!\shortmid & \end{array}$$

Cheeger–Chern–Simons–Weil

Given $(E, \nabla) \in \mathrm{Bun}_{U,\nabla}(M)$

$$\begin{array}{ccc} (E, \nabla) & & \text{Hermitian vector bundle with connection} \\ \downarrow & & \\ M & & \text{smooth manifold} \end{array}$$

Obtain

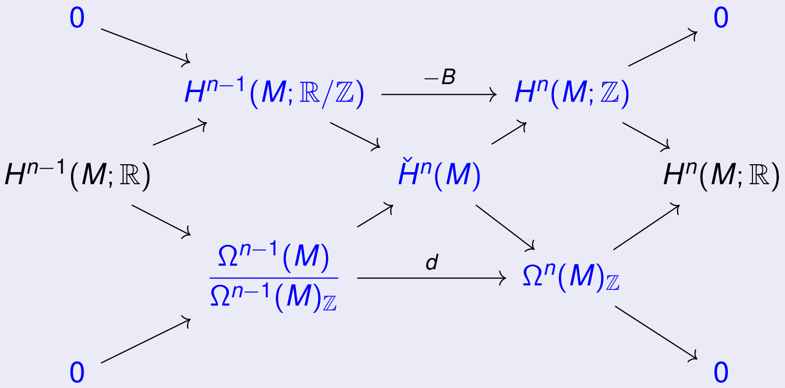
$$\begin{array}{ccccccc} \check{c}_k(E,\nabla) & \in & \check{H}^{2k}(M) & \longrightarrow & H^{2k}(M;\mathbb{Z}) & \ni & c_k(E) \\ & & \downarrow & & \downarrow & & \\ & & \Omega^{2k}(M)_{\text{closed}} & \longrightarrow & H^{2k}(M;\mathbb{R}) & & \\ & & \cup & & & & \\ & & c_k(\nabla) & \longmapsto & \bullet & & \end{array}$$

$$\check{c}_k(E,\nabla) \mapsto c_k(E), \quad \check{c}_k(E,\nabla) \mapsto c_k(\nabla), \quad c_k(E) \mapsto \bullet$$

Differential cohomology

Definition/Theorem (Cheeger–Simons)

Functors $\check{H}^*\colon \mathrm{Man}^{\mathrm{op}} \to \mathrm{AbelianGroups}$ satisfying:

- 

  where diagonals are short exact sequences,
- Chern–Weil homomorphism factors through $\check{H}^*$.

Equivariant cohomology

G compact Lie group acting on smooth manifold M

Borel

$$H^n_G(M;-) := H^n(EG \times_G M;-)$$

Weil/Cartan de Rham models

$$\Omega_G(M) := \left(S\mathfrak{g}^* \otimes \Lambda\mathfrak{g}^* \otimes \Omega(M)\right)_{\text{basic}} \cong \left(S\mathfrak{g}^* \otimes \Omega(M)\right)^G$$
$$|S^1\mathfrak{g}^*| = 2, \quad |\Lambda^1\mathfrak{g}^*| = 1$$

Chern–Weil

Given $(E, \nabla) \in \mathrm{Bun}_{U,\nabla}(M)$

$$\begin{array}{ccc} (E,\nabla) & & \text{Hermitian vector bundle with connection} \\ \downarrow & & \\ M & & \text{smooth manifold} \end{array}$$

Obtain

$$\begin{array}{ccccc} & & H^{2k}(M;\mathbb{Z}) & \ni & c_k(E) \\ & & \downarrow & & \\ \Omega^{2k}(M)_{\text{closed}} & \longrightarrow & H^{2k}(M;\mathbb{R}) & & \\ \cup\!\!\!| & & & & \\ c_k(\nabla) & \longmapsto & \bullet & \longleftarrow\!\shortmid & c_k(E) \end{array}$$

Equivariant Chern–Weil (Borel, Berline–Vergne)

G compact Lie group

Given $(E, \nabla) \in G\text{-Bun}_{U,\nabla}(M)$

$$\begin{array}{c} (E,\nabla) \\ \downarrow \\ M \end{array}$$

(E,∇): G-equivariant Hermitian vector bundle with G-invariant connection on

M: G-manifold

Obtain

$$\begin{array}{ccccc} & & H^{2k}_G(M;\mathbb{Z}) & \ni & c_k(E_G) \\ & & \downarrow & & \\ \Omega^{2k}_G(M)_{\text{closed}} & \longrightarrow & H^{2k}_G(M;\mathbb{R}) & & \\ \Cup & & & & \\ c_k(\nabla_G) & \longmapsto & \bullet & \leftarrow\!\shortmid & c_k(E_G) \end{array}$$

Definition/Theorem (R. and Kübel independently)

Functors $\check{H}^*_G \colon G\text{-}\mathrm{Man}^{\mathrm{op}} \to \mathrm{AbelianGroups}$ satisfying:

- $$\begin{array}{ccccccccc}
0 & & & & & & & & 0 \\
& \searrow & & & & & & \nearrow & \\
& & H^{n-1}_G(M;\mathbb{R}/\mathbb{Z}) & & \xrightarrow{-B_G} & & H^n_G(M;\mathbb{Z}) & & \\
& \nearrow & & \searrow & & \nearrow & & \searrow & \\
H^{n-1}_G(M;\mathbb{R}) & & & & \check{H}^n_G(M) & & & & H^n_G(M;\mathbb{R}) \\
& \searrow & & \nearrow & & \searrow & & \nearrow & \\
& & \dfrac{\Omega^{n-1}_G(M)}{\Omega^{n-1}_G(M)_{\mathbb{Z}}} & & \xrightarrow{d_G} & & \Omega^n_G(M)_{\mathbb{Z}} & & \\
& \nearrow & & & & & & \searrow & \\
0 & & & & & & & & 0
\end{array}$$

  where diagonals are short exact sequences,
- Chern–Weil homomorphism factors through $\check{H}^*_G$.

Key Idea

$\check{H}^*_G(M)$ is the differential cohomology of the *differential quotient stack* $\mathcal{E}_\nabla G \times_G M \cong [M /\!\!/_\nabla G]$

Goal for the remainder of the talk

Explain what this means and why it is natural from a differential geometric perspective.

Stacks – things that naturally pullback

Definitions

- A *groupoid* is a category whose morphisms are all isomorphisms.
- Any set is naturally a groupoid, with only identity morphisms.
- A *stack* $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Gpd}}$ is a "sheaf of groupoids on the site of manifolds;" a functor $\mathrm{Man}^{\mathrm{op}} \to \mathrm{Gpd}$ satisfying a sheaf condition.

Examples

- $M \in \mathrm{Man} \rightsquigarrow M \in \mathrm{Shv}_{\mathrm{Gpd}}$, by $M(X) := C^\infty(X, M) \in \mathrm{Set} \subset \mathrm{Gpd}$
- $\Omega^n \in \mathrm{Shv}_{\mathrm{Gpd}}$, by $\Omega^n(X) := \Omega^n(X) \in \mathrm{Set}$
- $\mathcal{B}_\nabla G(X) := \mathrm{Bun}_{G,\nabla}(X) =$ principal G-bundles with connection

$$\begin{cases} \text{Objects: } & \begin{array}{ccc} G & \hookrightarrow & (P, \Theta) \\ & & \downarrow \\ & & X \end{array} \\ \text{Morphisms: bundle isomorphisms preserving connection} & \end{cases}$$

Differential Quotient Stack

$M \in G$-Man

$$(\mathcal{E}_\nabla G \times_G M)(X) = \begin{cases} \text{Obj: } \begin{array}{ccc} (P,\Theta) & \xrightarrow{f} & M \\ \downarrow & & \\ X & & \end{array} & f \text{ is } G\text{-equivariant} \\ \text{Mor: preserve } \Theta, \text{ compatible with } f & \end{cases}$$

For $M = \text{pt}$, then $\mathcal{E}_\nabla G \times_G \text{pt} = \mathcal{B}_\nabla G$.

Maps between stacks

Yoneda

$$\text{“Map}(M, \mathcal{F})\text{”} = \mathrm{Shv}_{\mathrm{Gpd}}(M, \mathcal{F}) \cong \mathcal{F}(M)$$

Examples

- Stack maps generalize smooth maps between manifolds

$$\begin{array}{ccccc} & & \mathrm{Man}(M,N) & = & C^\infty(M,N) \\ \mathrm{Map}(M,N) & \begin{array}{c} \overset{?}{\cong} \\ \overset{?}{\cong} \end{array} & & & \| \\ & & \mathrm{Shv}_{\mathrm{Gpd}}(M,N) & \cong & N(M) \end{array}$$

- $\omega \in \Omega^n(M)$ is equivalent to $M \xrightarrow{\omega} \Omega^n$
- $(P, \Theta) \in \mathrm{Bun}_{G,\nabla}(M)$ is equivalent to $M \xrightarrow{(P,\Theta)} \mathcal{B}_\nabla G$

Important special case: $M = \text{pt}$

Differential Forms

$\Omega^n(\mathcal{B}_\nabla G) := ?\ \text{Map}(\mathcal{B}_\nabla G, \Omega^n)$

What is a differential form $\omega \in \Omega^n(\mathcal{B}_\nabla G)$?

$$\begin{array}{ccc} \mathcal{B}_\nabla G(X) & \xrightarrow{\omega} & \Omega^n(X) \\ {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow \\ \mathcal{B}_\nabla G(Y) & \xrightarrow{\omega} & \Omega^n(Y) \end{array}$$

In other words,

- to any $(P, \Theta) \to X$, we get $\omega(P, \Theta) \in \Omega^n(X)$,
- if $(P, \Theta) \cong (P', \Theta')$, then $\omega(P, \Theta) = \omega(P', \Theta')$,
- $\omega(f^*(P, \Theta)) = f^*(\omega(P, \Theta))$.

Important special case: $M = \text{pt}$ (continued)

Differential Forms

$\Omega^n(\mathcal{B}_\nabla G) = \Omega^n(\mathcal{B}_\nabla G) := \text{Map}(\mathcal{B}_\nabla G, \Omega^n)$

Observation

Chern–Weil construction gives homomorphism

$$\Omega_G(\text{pt}) = (S\mathfrak{g}^*)^G \xrightarrow{\cong} \Omega(\mathcal{B}_\nabla G) = \Omega(\mathcal{E}_\nabla G \times_G \text{pt}).$$

Theorem (Freed–Hopkins)

This map is an *isomorphism*.

"Goldilocks"

- $\Omega(BG)$ - too big
- $\Omega(\mathcal{B}G)$ - too small ($\Omega^n(\mathcal{B}G) = 0$ for $n > 0$)
- $\Omega(\mathcal{B}_\nabla G)$ - just right

Equivariant forms: General case

Given

$$\begin{array}{ccc} (P,\Theta) & \xrightarrow{f} & M \\ \downarrow & & \\ X & & \end{array} \quad , \quad \omega = \alpha\beta\gamma \in \left(S^i\mathfrak{g}^* \otimes \Lambda^j\mathfrak{g}^* \otimes \Omega^k(M)\right)_{\text{basic}}$$

Obtain

$$\alpha(\Omega^{\wedge i}) \wedge \beta(\Theta^{\wedge i}) \wedge f^*\gamma \in \Omega(P)_{\text{basic}} \cong \Omega(X)$$

Theorem (Freed–Hopkins)

$$\Omega_G(M) \xrightarrow{\cong} \Omega(\mathcal{E}_\nabla G \times_G M).$$

Differential equivariant cohomology

Definition: ($M \in G$-Man)

$\check{H}^n_G(M) := \check{H}^n(\mathcal{E}_\nabla G \times_G M) := \mathrm{Map}(\mathcal{E}_\nabla G \times_G M, \check{H}^n)$
(some details being ignored here)

In other words, an element $\check{\lambda} \in \check{H}^n_G(M)$ is a construction:

- to any $\begin{array}{ccc} (P,\Theta) & \xrightarrow{f} & M \\ \downarrow & & \\ X & & \end{array}$ associate $\check{\lambda}(P,\Theta,f) \in \check{H}^n(X)$,
- if $\begin{array}{ccccc} (P_1,\Theta_1) & \xrightarrow{\varphi} & (P_2,\Theta_2) & \xrightarrow{f} & M \\ \downarrow & & \downarrow & & \\ X_1 & \xrightarrow{\overline{\varphi}} & X_2 & & \end{array}$, then
$\check{\lambda}(P_1,\Theta_1, f\circ\varphi) = \overline{\varphi}^*\check{\lambda}(P_2,\Theta_2,f) \in \check{H}^n(X_1)$.

Differential Equivariant Cohomology (continued)

Abbreviate $\mathcal{E}_\nabla G \times_G M$ by $M/\!\!/_\nabla G$,

Theorem (R.)

The following diagram ...

$$
\begin{array}{ccccccccc}
0 & & & & & & & & 0 \\
& \searrow & & & & & & \nearrow & \\
& & H^{n-1}(M/\!\!/_\nabla G;\mathbb{R}/\mathbb{Z}) & & \xrightarrow{-B} & & H^n(M/\!\!/_\nabla G;\mathbb{Z}) & & \\
& \nearrow & & \searrow & & \nearrow & & \searrow & \\
H^{n-1}(M/\!\!/_\nabla G;\mathbb{R}) & & & & \check{H}^n(M/\!\!/_\nabla G) & & & & H^n(M/\!\!/_\nabla G;\mathbb{R}) \\
& \searrow & & \nearrow & & \searrow & & \nearrow & \\
& & \dfrac{\Omega^{n-1}(M/\!\!/_\nabla G)}{\Omega^{n-1}(M/\!\!/_\nabla G)_{\mathbb{Z}}} & & \xrightarrow{d} & & \Omega^n(M/\!\!/_\nabla G)_{\mathbb{Z}} & & \\
& \nearrow & & & & & & \searrow & \\
0 & & & & & & & & 0
\end{array}
$$

Differential Equivariant Cohomology (continued)

Abbreviate $\mathcal{E}_{\nabla}G \times_G M$ by $M/\!\!/_{\nabla}G$,

Theorem (R.)

...naturally isomorphic to

$$
\begin{array}{ccccccccc}
0 & & & & & & & & 0 \\
& \searrow & & & & & & \nearrow & \\
& & H^{n-1}_G(M;\mathbb{R}/\mathbb{Z}) & & \xrightarrow{-B_G} & & H^n_G(M;\mathbb{Z}) & & \\
& \nearrow & & \searrow & & \nearrow & & \searrow & \\
H^{n-1}_G(M;\mathbb{R}) & & & & \check{H}^n_G(M) & & & & H^n_G(M;\mathbb{R}) \\
& \searrow & & \nearrow & & \searrow & & \nearrow & \\
& & \dfrac{\Omega^{n-1}_G(M)}{\Omega^{n-1}_G(M)_{\mathbb{Z}}} & & \xrightarrow{d_G} & & \Omega^n_G(M)_{\mathbb{Z}} & & \\
& \nearrow & & & & & & \searrow & \\
0 & & & & & & & & 0
\end{array}
$$

"Equivariant Cheeger–Chern–Simons"

Theorem (R.)

- There is a natural equivalence of categories $\{G\text{-equivariant (Herm vector bundles w/ conn on } M)\}$ $\cong \mathrm{Map}(\mathcal{E}_\nabla G \times_G M, \mathcal{B}_\nabla U)$
- Given $(E, \nabla) \in G\text{-Bun}_{U,\nabla}(M)$,

$$\mathcal{E}_\nabla G \times_G M \xrightarrow{(E,\nabla)_G} \mathcal{B}_\nabla U$$

$$H^*_G(M) = \check{H}^*(\mathcal{E}_\nabla G \times_G M) \xleftarrow{(E,\nabla)^*_G} \check{H}^*(\mathcal{B}_\nabla U)$$

$$\check{c}_k(E_G, \nabla_G) \longleftarrow\!\shortmid \quad \check{c}_k$$

refines the equivariant Chern–Weil homomorphism.