

CANONICAL METRIC CONNECTIONS ASSOCIATED TO STRING
STRUCTURES

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Abstract

by

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In this thesis, we look at principal $Spin(n)$ -bundles $P \rightarrow M$ whose Pontryagin class $\frac{p_1}{2}(P) = 0 \in H^4(M; \mathbb{R})$. We say that such a bundle admits a string structure, and a choice of string structure is given by particular elements in $H^3(P; \mathbb{Z})$. This provides an analogue to the idea of a spin structure, where the topological group $String(n)$ is, up to homotopy, the unique 3-connected cover of $Spin(n)$ ($n \geq 5$). Choosing a Riemannian metric on the base M and a connection on P determines a 1-parameter family of metrics on P . We prove that in a scaling limit known as the adiabatic limit, the harmonic representative of a string structure is equal to the Chern–Simons 3-form on P minus a 3-form on M , denoted $H \in \Omega^3(M)$. This 3-form H is closely related to the Chern–Simons form, and can be thought of as a reduction of the Chern–Simons form on P to a form on M . The exterior derivative of H is the $\frac{p_1}{2}$ -form; the integral of H on any 3-cycle is equal, modulo \mathbb{Z} , to the integral of the Chern–Simons 3-form pulled back via a global section on the same 3-cycle. Finally, we note that for the $Spin(n)$ -frame bundle $Spin(M)$ of a Riemannian spin manifold, this 3-form H determines a canonical metric connection on M . This canonical metric connection depends on both the metric and string structure, and its torsion is determined by H .

To Mom and Dad.

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CHAPTER 1

INTRODUCTION

1.1 Summary

The Pontryagin class $\frac{p_1}{2}(M) \in H^4(M; \mathbb{Z})$, defined on a closed compact spin manifold M , is a common obstruction to defining (even formally) certain operators or constructions in quantum field theory. For example, the spinor bundle on the loop space LM exists if $\frac{p_1}{2}(M) = 0$, but not in general [CP]. Also, important 2-dimensional supersymmetric sigma models associated to a Riemannian manifold with spin structure don't even make formal sense unless $\frac{p_1}{2}(M) = 0$ [AS1].

The definition of a string structure is not a universal one, but in this thesis, a spin manifold M admits a string structure if and only if $\frac{p_1}{2}(M) = 0$. Given a spin structure on M , let $Spin(M) \xrightarrow{\pi} M$ be the $Spin(n)$ -frame bundle (where $n \geq 5$). A string structure is then defined as a cohomology class $\mathcal{S} \in H^3(Spin(M); \mathbb{Z})$ that restricts to the (standard) generator of $H^3(Spin(n); \mathbb{Z}) \cong \mathbb{Z}$ on the fibers. The primary goal of this thesis is the following construction. Given a Riemannian spin manifold (M, g) , there is a natural 1-parameter family of metrics g_δ on the bundle $Spin(M)$, and the limit as $\delta \rightarrow 0$ is referred to as the adiabatic limit. Given a string structure $\mathcal{S} \in H^3(Spin(M); \mathbb{Z})$, we take the harmonic representative in the adiabatic limit of (the real cohomology class given by) \mathcal{S} . By Theorem 6.4.4, this form is equal to the Chern–Simons 3-form $\alpha(\Theta)$ minus a 3-form $H_{g, \mathcal{S}}$ pulled back

from M ; i.e.

$$\lim_{\delta \rightarrow 0} [\mathcal{S}]_{g\delta} = \alpha(\Theta) - \pi^* H_{g,S} \in \Omega^3(P).$$

The form $H_{g,S} \in \Omega^3(M)$ is closely related to the Chern–Simons 3-form on P , and can be thought of as a reduction of the Chern–Simons 3-form to a form on M . In particular, $dH_{g,S} = \frac{p_1}{2}(M, g)$, and the harmonic component of $H_{g,S}$, once reduced to $H^3(M; \mathbb{R}/\mathbb{Z})$, equals the 3-dimensional Chern–Simons cohomology class. The forms $\alpha(\Theta) - \pi^* H_{g,S}$ and $H_{g,S}$ arise in both the construction of the spinor bundle on LM and making formal sense of 2-dimensional supersymmetric sigma models with target space M . In future work, the author hopes to formalize the link between these constructions and the results in this thesis.

Given a choice of metric and string structure on M , we have the canonical 3-form $H_{g,S}$, and we use this to construct a canonical 1-parameter family of metric connections $\nabla^{\epsilon g, S}$ on TM . These connections are characterized their torsion, which is given by

$$\langle T^{\epsilon g, S}(X, Y), Z \rangle_g = \frac{1}{\epsilon} H_{g,S}(X, Y, Z).$$

The limit as $\epsilon \rightarrow \infty$ gives the Levi-Civita connection, and the limit as $\epsilon \rightarrow 0$ is not a connection for $H_{g,S} \neq 0$. Connections with torsion have become important in string theory. Though there are similarities between this family of connections and other connections appearing in the literature, the precise relationships are not yet known by the author. The hope is that geometric properties of this family of connections is related to invariants coming from conformal field theories and elliptic cohomology. This motivated the development of these connections, but there is not yet any mathematical link. We will go ahead and briefly describe the possible relationship, but it should be noted that this is only one possible use of the results in this thesis.

1.2 Original motivation

In [Sto], Stolz conjectures that if a manifold M admits both a metric of positive Ricci curvature and a string structure, then a topological invariant called the Witten genus vanishes; i.e. $\phi_W(M) = 0$. While this remains a valid conjecture, the naive guess that positive Ricci curvature could imply the vanishing of the refined Witten genus, which lives in the cohomology theory tmf , is simply not true. The hope is that some variant of this statement holds true when considering the curvatures of the canonical 1-parameter family of connections depending on both the metric and the string structure.

There is a simpler but analogous situation. If a spin manifold M admits a metric of positive scalar curvature $s > 0$, then $\widehat{A}(M) = 0$ [Lic]. This can be seen by the Atiyah-Singer Index Theorem and the Bochner-Weizenböck-Lichnerowicz formula $\mathcal{D}^2 = \nabla^* \nabla + \frac{s}{4}$ [LM]. If $s > 0$, then \mathcal{D}^2 is a strictly positive operator, and hence its kernel and $\widehat{A}(M)$ are both 0. There is also a refinement of \widehat{A} given by the spin-orientation of KO -theory. This induces a map $\alpha : \Omega_n^{Spin}(pt) \rightarrow KO^{-n}(pt)$, which depends on the choice of spin structure, such that the following diagram commutes:

$$\begin{array}{ccc}
 & & KO^{-n}(pt) \\
 & \nearrow \alpha & \downarrow \\
 \Omega_n^{Spin}(pt) & \xrightarrow{\widehat{A}} & \mathbb{Z}.
 \end{array}$$

For a spin structure d , the element $\alpha[M, d] \in KO^{-n}(pt)$, sometimes called the Atiyah α -invariant, can be thought of as the Clifford-linear index of \mathcal{D}_M . The above construction usually appears in family index theorems, but it also contains interesting information for a single manifold due to the torsion in $KO^{-*}(pt)$. By a theorem of Hitchin, one can use a similar proof to show that if M admits a positive scalar curvature metric, then $\alpha[M, d] = 0 \in KO^{-n}(pt)$ for *any* spin structure d [Hit].

This result has a nice interpretation within the context of quantum field theory. Stolz and Teichner have shown that $KO^{-n} \simeq \mathcal{EFT}_n$, i.e. that the $(-n)$ -th space in the KO -theory spectrum is homotopy equivalent to the space of supersymmetric 1-dimensional Euclidean Field Theories of degree n [ST]. Roughly speaking, one can interpret the map α as, up to homotopy, quantizing fermionic particles moving in a target spin manifold M . The integer $\widehat{A}(M)$ is the partition function of the resulting field theory. Hitchin's theorem implies that if M^n admits a positive scalar curvature metric, the 1-dimensional field theory associated to M^n is qualitatively the same as the trivial field theory of degree n . From the perspective of Stolz and Teichner, the relationship between KO -theory and 1-dimensional Euclidean field theories should be analogous to the relationship between 2-dimensional conformal field theories and elliptic cohomology.

As mentioned above, Stolz conjectures that the following analogous situation should hold for string manifolds: if M admits a string-structure and a metric of positive Ricci curvature, then the Witten genus $\phi_W(M) = 0$ [Sto]. His heuristic argument comes from interpreting the Witten genus as the S^1 -equivariant index of the Dirac operator on the free loop space LM (i.e. $\phi_W(M) = \text{index}^{S^1}(\not{D}_{LM})$) [Wit2][Wit3]. Also, there should be some Weizenböck-type formula involving $Ric(M)$ such that if $Ric(M) > 0$, then $\text{Ker}(\not{D}_{LM}) = 0$. This line of argument is far from rigorous since \not{D}_{LM} and the scalar curvature on LM are not well-defined mathematical objects. However, the conjecture holds true for homogeneous spaces and complete intersections. Currently, there are no known examples of simply connected manifolds which admit metrics of positive scalar curvature, but not metrics of positive Ricci curvature. If the conjecture is true, it would provide examples of such manifolds.

Just as the cohomology theory KO is the home for the \widehat{A} -genus of a spin manifold

(or a family of spin manifolds), there is a cohomology theory tmf (topological modular forms) that is the natural home for the Witten genus ϕ_W of a string manifold ([Hop], [AHS]). In other words, there is a map σ such that the following diagram commutes:

$$\begin{array}{ccc}
 & & tmf^{-n}(pt) \\
 & \nearrow \sigma & \downarrow \\
 \Omega_n^{String}(pt) & \xrightarrow{\phi_W} & MF_n
 \end{array}$$

The map $tmf^{-n}(pt) \rightarrow MF_n$ is a rational isomorphism, where MF_n denotes modular forms of weight $\frac{n}{2}$. There are several attempts to geometrically define tmf (or other elliptic cohomology theories) in terms of 2-dimensional conformal field theories (including [Seg1], [Seg2], and [ST]). Under this view, $\sigma(M)$ is (up to homotopy) the super-symmetric non-linear sigma model for a target space M . The partition function would be $\phi_W(M)$.

It is important to note that both $\widehat{A}(M)$ and $\phi_W(M)$ are topologically defined and thus independent of spin or string structure. However, the maps α and σ into the appropriate cohomology theories do depend on such choices. Now, one might hope that the analog of Hitchin's theorem holds. In other words, is it possible that if M^n admits a metric of positive Ricci curvature, then $\sigma[M^n, \mathcal{S}] = 0 \in tmf^{-n}(pt)$ for all string structures \mathcal{S} ? The answer is no, and counterexamples can be found by looking at Lie groups.

Assume G is a compact, semi-simple Lie group. G admits a bi-invariant metric which has positive Ricci (and sectional) curvature. G also has two canonical string structures defined by the left and right invariant framing of TG , which we will denote \mathcal{L} and \mathcal{R} . The image of these under σ has been computed in many examples and are often not 0 [Hop], hence the above question cannot have an affirmative answer. The easiest case to see is $SU(2) \cong S^3$. In addition to the string structures coming

from left and right invariant framing, there is the trivial string structure given by $S^3 = \partial D^4$. We then have the following:

$$\begin{aligned}\Omega_3^{String} &\xrightarrow{\sigma} tmf^{-3} \cong \mathbb{Z}/24 \\ [SU(2), \mathcal{L}] &\longmapsto \frac{-1}{24} \\ [SU(2), \partial D^4] &\longmapsto 0 \\ [SU(2), \mathcal{R}] &\longmapsto \frac{1}{24}\end{aligned}$$

The above follows from the fact that the maps $\pi_3 S^0 \rightarrow \pi_3 MString \rightarrow \pi_3 tmf$ are all isomorphisms [Hop]. This reduces to a framed bordism calculation, and using the Adams e -invariant, one can show that the different framings on $SU(2)$ give the above framed bordism classes [AS2].

Using Proposition 6.5.1 and the $H^3(M; \mathbb{Z})$ -equivariance of our construction, we see that for the bounding string structure ∂D^4 , the 3-form given by our construction is

$$H_{g, \partial D^4} = 0 \in \Omega^3(SU(2)).$$

Therefore, the resulting 1-parameter family of connections $\nabla^{\epsilon g, \partial D^4}$ is always the Levi-Civita connection, and hence always has positive curvature. However, the 1-parameter family of connections $\nabla^{\epsilon g, \mathcal{L}}$, $\nabla^{\epsilon g, \mathcal{R}}$ have negative Ricci curvature for ϵ small. From the description of the Ricci curvature in (7.0.1), we see that in general, if $H_{g, \mathcal{S}} \neq 0$, then $Ric \nabla^{\epsilon g, \mathcal{S}} \rightarrow -\infty$ as $\epsilon \rightarrow 0$. The most obvious question is: if $Ric \nabla^{\epsilon g, \mathcal{S}} > 0$ for all ϵ , does this imply that $\sigma(M, \mathcal{S}) = 0 \in tmf^{-n}(pt)$? Equivalently, if (M, g) has positive Ricci curvature and $H_{g, \mathcal{S}} = 0$, does this imply that $\sigma(M, \mathcal{S}) = 0$?

1.3 Outline of the thesis

We now proceed to give a more in-depth outline of the actual thesis. Chapters 2, 3, and 4 all build up to the calculation in Chapter 5 of the 3-dimensional harmonic forms on a principal G -bundle (G simple) in the adiabatic limit. This calculation leads to the canonical 3-forms on M associated to a metric and string structure (and connection). Finally, the canonical 3-forms give rise to the canonical connections in Chapter 7.

All manifolds are considered to be closed and compact. Let $P \xrightarrow{\pi} M$ be a principal G -bundle, for G a compact Lie group. We begin in Chapter 2 by reviewing the theory of connections on principal bundles, the basics of Hodge theory, as well as the Chern–Simons and Chern–Weil forms we use. On any principal bundle, there is a canonical distribution of “vertical” vectors given by

$$T^V P \stackrel{\text{def}}{=} \text{Ker}(\pi_*) \subset TP.$$

A choice of connection is equivalent to the choice of an equivariant distribution of “horizontal” vectors. Letting Θ denote the connection 1-form on P , this is written as

$$T^H P \stackrel{\text{def}}{=} \text{Ker} \Theta \subset TP.$$

Therefore, a connection induces a bi-grading on the space of differential forms, denoted

$$\Omega^{i,j}(P) \stackrel{\text{def}}{=} C^\infty(P, \Lambda^i T^H P^* \otimes \Lambda^j T^V P^*).$$

This bi-grading is extremely helpful in later calculations. A connection also induces Chern–Weil forms on M and Chern–Simons forms on P . In particular, for G simple, the Chern–Simons 3-form is denoted $\alpha(\Theta) \in \Omega^3(P)$, and its derivative is the pullback of the Chern–Weil form $\langle \Omega \wedge \Omega \rangle \in \Omega^4(M)$ (where $\langle \cdot, \cdot \rangle$ is a suitably normalized Ad -

invariant metric on \mathfrak{g} , and Ω is the curvature 2-form).

By definition, G acts freely from the right on P . If the metric on P is G -equivariant, then the harmonic forms will be G -invariant. Therefore, in Chapter 3, we give a useful description of the subcomplex of right-invariant differential forms on P . In addition to aiding calculations, the author finds this description conceptually helpful; it relates the exterior derivative on P to the Lie algebra derivative, the connection Θ , and the curvature Ω .

The description is given by an isomorphism of cochain complexes. To see this, first note that there is a canonical isomorphism of G -equivariant vector bundles

$$T^V P \cong P \times \mathfrak{g} = \pi^* \mathfrak{g}_P,$$

where $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$ is the adjoint vector bundle over M . Given a connection Θ ,

$$T^H P = \pi^* TM,$$

and thus we have a canonical isomorphism

$$\Omega^{i,j}(P)^G \xleftarrow{\sim} \Omega^i(M; \Lambda^j \mathfrak{g}_P^*).$$

The exterior derivative on $\Omega^{i,j}(P)$ decomposes with respect to the bi-grading, and

$$d = d^{0,1} + d^{1,0} + d^{2,-1}$$

where $d^{a,b} : \Omega^{i,j}(P) \rightarrow \Omega^{i+a,j+b}(P)$. When restricted to right-invariant forms, the derivations $d^{a,b}$ are given by familiar maps on the isomorphic complex $\Omega^i(M; \Lambda^j \mathfrak{g}_P^*)$. The derivative of right-invariant forms on the fibers induces the vector bundle homomorphism

$$\Lambda^j \mathfrak{g}_P^* \xrightarrow{d_{\mathfrak{g}}} \Lambda^{j+1} \mathfrak{g}_P^*.$$

This gives the map

$$d^{0,1} = (-1)^i d_{\mathfrak{g}} : \Omega^i(M; \Lambda^j \mathfrak{g}_P^*) \rightarrow \Omega^i(M; \Lambda^{j+1} \mathfrak{g}_P^*).$$

The principal bundle connection Θ induces a vector bundle connection ∇ on each associated vector bundle $\Lambda^j \mathfrak{g}_P^*$, and

$$d^{1,0} = d_{\nabla} : \Omega^i(M; \Lambda^j \mathfrak{g}_P^*) \rightarrow \Omega^{i+1}(M; \Lambda^j \mathfrak{g}_P^*).$$

Finally, the curvature of a connection is an element $\Omega \in \Omega^2(M; \mathfrak{g}_P)$, and $d^{2,-1}$ is given by contracting along this vector-valued 2-form; this is denoted

$$d^{2,-1} = (-1)^i \iota_{\Omega} : \Omega^i(M; \Lambda^j \mathfrak{g}_P^*) \rightarrow \Omega^{i+2}(M; \Lambda^{j-1} \mathfrak{g}_P^*).$$

In particular, $d^{2,-1} = 0$ if and only if the connection is flat.

To make sense of Hodge cohomology, P must be a Riemannian manifold. Given a Riemannian metric g on M and a connection on P , we see a natural family of metrics on P . First, choose a bi-invariant metric g_G on G , or equivalently, choose an Ad -invariant metric on \mathfrak{g} . Such metrics exist because G is compact, and all theorems in Chapter 5 are independent of this choice. Defining the vertical and horizontal spaces to be perpendicular then gives the metric g_P , written

$$g_P \stackrel{\text{def}}{=} \pi^*(g \oplus g_G)$$

under the decomposition $TP \cong \pi^*(TM \oplus \mathfrak{g}_P)$.

The main result of this thesis follows from the 3-dimensional Hodge cohomology of P . However, these forms only have a nice description under the scaling limit given by shrinking the fibers relative to the base, or enlarging the base relative to the fibers. Introducing this scaling factor δ , define

$$g_{\delta} \stackrel{\text{def}}{=} \pi^*(\delta^{-2}g \oplus g_G).$$

The limit $\delta \rightarrow 0$ is referred to as the adiabatic limit, and was introduced by Witten in [Wit1]. It has produced a number of interesting results, including [BF]. Though g_0 is not a metric, the work of [MM] and [For] gives a smooth extension of $\text{Ker } \Delta_{g_\delta}$ to $\delta = 0$. The goal of Chapter 5 is to describe

$$\lim_{\delta \rightarrow 0} \text{Ker } \Delta_{g_\delta}^3 \subset \Omega^3(P).$$

Note that for any $\delta > 0$, g_δ is a right-invariant Riemannian metric on P , and is given by the pullback of an inner product on $\Omega^i(M; \Lambda^j \mathfrak{g}_P^*)$. The adjoint d^{*g_δ} on $\Omega^*(P)$, when restricted to right-invariant forms, then decomposes in terms of the adjoints of $d_{\mathfrak{g}}$, d_{∇} , and ι_Ω .

Chapter 4 summarizes results concerning a Hodge-theoretic version of the Leray–Serre spectral sequence. This spectral sequence first appeared in [MM] and was later made explicit in [For]. A key role is played by the isometry

$$\begin{aligned} \rho_\delta : (\Omega^{i,j}(P), g_\delta) &\longrightarrow (\Omega^{i,j}(P), g_P) \\ \phi &\longmapsto \delta^i \phi. \end{aligned}$$

In addition to fixing the inner product space, this also induces factors of δ which give the structure of a spectral sequence. Conjugating by the isometry ρ_δ gives a rescaled derivative d_δ , a rescaled coderivative d_δ^* , and a rescaled Laplacian L_{g_δ} .

$$\begin{aligned} d_\delta &\stackrel{\text{def}}{=} \rho_\delta d \rho_\delta^{-1} = d^{0,1} + \delta d^{1,0} + \delta^2 d^{2,-1}, \\ d_\delta^* &\stackrel{\text{def}}{=} \rho_\delta d^{*g_\delta} \rho_\delta^{-1} = d^{0,1} + \delta d^{1,0} + \delta^2 d^{2,-1}, \\ L_{g_\delta} &\stackrel{\text{def}}{=} \rho_\delta \Delta_{g_\delta} \rho_\delta^{-1} = d_\delta d_\delta^* + d_\delta^* d_\delta. \end{aligned}$$

When restricted to right-invariant forms, d_δ and d_δ^* take the form

$$\begin{aligned} d_\delta &= \pm d_{\mathfrak{g}} + \delta d_\nabla \pm \delta^2 \iota_\Omega, \\ d_\delta^* &= \pm d_{\mathfrak{g}}^* + \delta d_\nabla^* \pm \delta^2 \iota_\Omega^*. \end{aligned}$$

Treating δ as a formal variable, there is a filtration of forms on P given by

$$\begin{aligned} E_K^{i,j} &= \{ \omega \in \Omega^{i,j}(P) \mid \exists \omega_1, \dots, \omega_l \text{ with} \\ & d_\delta(\omega + \delta\omega_1 + \dots + \delta^l \omega_l) \in \delta^K \Omega^*(P)[\delta] \\ & d_\delta^*(\omega + \delta\omega_1 + \dots + \delta^l \omega_l) \in \delta^K \Omega^*(P)[\delta] \}. \end{aligned}$$

This forms a spectral sequence which is isomorphic to the Leray–Serre spectral sequence for the fibration $G \hookrightarrow P \xrightarrow{\pi} M$. In addition to the abstract convergence $E_\infty^k \cong H^k(P; \mathbb{R})$ (where $k = i + j$) given by the Leray–Serre spectral sequence, there is the geometric/analytic statement that

$$L_0^k \stackrel{\text{def}}{=} \lim_{\delta \rightarrow 0} \text{Ker } L_{g_\delta}^k = E_\infty^k.$$

In fact, the space L_0^k is a *smooth* extension of $\text{Ker } L_{g_\delta}^k$ to $\delta = 0$. This also gives a smooth extension of $\text{Ker } \Delta_{g_\delta}^k$ to $\delta = 0$. If we define the formal rescaled Laplacian

$$L_\delta^k : \Omega^p(P)[[\delta]] \longrightarrow \Omega^p(P)[[\delta]],$$

then elements $\omega_\delta \in \text{Ker } L_\delta$ are given by the Taylor series at $\delta = 0$ of sections $C^\infty([0, 1], \text{Ker } L_{g_\delta}^k)$, where $\text{Ker } L_{g_\delta}^k$ is a finite-dimensional vector bundle over $[0, 1]$.

The fact that $\{E_K^{i,j}\}$ is isomorphic to the Leray–Serre spectral sequence allows us to write down the lower-order terms (with respect to δ) in a basis of $\text{Ker } L_\delta$. A priori, we would have to construct a power series that is formally harmonic. However, if

we have a polynomial $\omega_\delta = \omega + O(\delta) \in \Omega^k(P)[\delta]$ such that

$$d_\delta \omega_\delta \in \delta^{N(k)} \Omega^{k+1}(P)[\delta], \quad d_\delta^* \omega_\delta \in \delta^{N(k)} \Omega^{k-1}(P)[\delta],$$

where $N(k)$ is the term where the spectral sequence calculating $H^k(P; \mathbb{R})$ collapses, then we are guaranteed the existence of higher order terms making $\omega + O(\delta)$ formally L_δ -harmonic. In fact, the same argument implies that if we have

$$\omega_\delta = \omega + \delta \omega_1 + \cdots + \delta^k \omega_k + O(\delta^{k+1}) \in \Omega^k(P)[\delta]$$

such that

$$d_\delta \omega_\delta \in \delta^{N(k)+k} \Omega^{k+1}(P)[\delta], \quad d_\delta^* \omega_\delta \in \delta^{N(k)+k} \Omega^{k-1}(P)[\delta],$$

then there exists a power series of the form

$$\omega + \delta \omega_1 + \cdots + \delta^k \omega_k + O(\delta^{k+1}) \in \text{Ker } L_\delta.$$

Furthermore, if this power series is in the domain of ρ_δ^{-1} (acting on formal power series), then we see the following relationship with Δ_{g_δ} -harmonic forms in the adiabatic limit:

$$\omega^{0,k} + \omega_1^{1,k-1} + \cdots + \omega_k^{k,0} \in \text{Ker } \Delta_0^k = \lim_{\delta \rightarrow 0} \text{Ker } \Delta_{g_\delta}^k$$

where $\omega_l^{i,j}$ denotes the projection of ω_l onto $\Omega^{i,j}(P)$. This is the method by which we describe $\text{Ker } \Delta_0^k$ in calculations.

Chapter 5 is the technical heart of the thesis. Roughly speaking, we look at principal G -bundles where the first real characteristic class is 0 and prove that low-dimensional harmonic forms ($k = 1, 2, 3$) on P in the adiabatic limit have a natural description in terms of a Chern–Simons form on P and harmonic forms on (M, g) . This is usually done in two parts. First, we describe the lower-order terms of elements in $\text{Ker } L_\delta$. The vanishing of the relevant characteristic class implies that the relevant part of the spectral sequence collapses at $N = 2$. Therefore, we

only need to produce polynomials in $\Omega^*(P)$ such that d_δ and d_δ^* are of the order of δ^{2+k} (where $k = 1, 2, 3$). Our explicit description of the right-invariant forms on P , along with standard properties of the Chern–Simons forms, makes it possible to produce such polynomials. After describing $\text{Ker } L_\delta$, we use the isometry ρ_δ^{-1} to obtain elements of $\text{Ker } \Delta_0$. We frequently use the notation

$$\mathcal{H}_g^k(M) \stackrel{\text{def}}{=} \text{Ker } \Delta_g^k \subset \Omega^k(M)$$

to denote the harmonic k -forms on the base with respect to the metric g .

The first case is for a principal $U(n)$ -bundle P with connection Θ over a Riemannian manifold (M, g) . We denote this by (M, P, g, Θ) . If $c_1(P) = 0 \in H^2(M; \mathbb{R})$, then for any $\delta \geq 0$ (where g_G is any Ad -invariant metric on $u(n)$),

$$\text{Ker } \Delta_{g_\delta}^1 = \mathbb{R} \left[\frac{i}{2\pi} \text{Tr}(\Theta) - \pi^* h \right] \oplus \pi^* \mathcal{H}_g^1(M).$$

Here, $h \in \Omega^1(M)$ is the unique form (under the Hodge decomposition) such that $h \in d^* \Omega^2(M)$ and $dh = c_1(P, \Theta)$. The second case is for G semi-simple. Given (M, P, g, Θ) and any Ad -invariant metric g_G on \mathfrak{g} ,

$$\text{Ker } \Delta_{g_\delta}^1 = \pi^* \mathcal{H}_g^1(M) \quad (\forall \delta \geq 0),$$

$$\text{Ker } \Delta_0^2 = \pi^* \mathcal{H}_g^2(M).$$

These two cases are included primarily as a warm-up to the next case.

The main technical result of the thesis is Theorem 5.3.2. For G simple, we begin with a principal G -bundle P with connection Θ over a Riemannian manifold (M, g) . Letting Θ denote the curvature 2-form, we assume that the real characteristic class

$$[\langle \Omega \wedge \Omega \rangle] = 0 \in H^4(M; \mathbb{R})$$

(e.g. $\frac{p_1}{2}(P) = 0$ or $c_2(P) = 0$ in $H^4(M; \mathbb{R})$). Then, for any Ad -invariant g_G ,

$$\text{Ker } \Delta_0^3 = \lim_{\delta \rightarrow 0} \text{Ker } \Delta_{g_\delta} = \mathbb{R}[\alpha(\Theta) - \pi^*h] \oplus \pi^*\mathcal{H}_g^3(M).$$

Here, $\alpha(\Theta)$ is the Chern–Simons 3-form, and $h \in \Omega^3(M)$ is the unique form such that $h \in d^*\Omega^4(M)$ and $dh = \langle \Omega \wedge \Omega \rangle$.

In Chapter 6, we give a topological definition of a string structure. Note that (for $n \geq 5$)

$$\pi_3(\text{Spin}(n)) \cong H^3(\text{Spin}(n); \mathbb{Z}) \cong \mathbb{Z}.$$

Up to homotopy, we define the topological group $\text{String}(n)$ as the 3-connected cover of $\text{Spin}(n)$. The existence of such a group is given by [ST], but we do not need any other properties of these more explicit descriptions. For a $\text{Spin}(n)$ -bundle $P \xrightarrow{\pi} M$, there is a 1-1 correspondence between isomorphism classes of $\text{String}(n)$ -bundles covering P and cohomology classes $\mathcal{S} \in H^3(P; \mathbb{Z})$ such that $i^*\mathcal{S} \in H^3(\text{Spin}(n); \mathbb{Z})$ is the (preferred) generator. We define string structures as these cohomology classes. A string structure exists if and only if $\frac{p_1}{2}(P) = 0 \in H^4(M; \mathbb{Z})$, and the space of string structures is a torsor for $H^3(M; \mathbb{Z})$. This is actually discussed for more general situations, and then specialized to define spin , \tilde{U} , spin^c , and string structures.

This leads to the following construction: given a principal $\text{Spin}(n)$ -bundle P with connection Θ over (M, g) , along with a string structure $\mathcal{S} \in H^3(P; \mathbb{Z})$, take the harmonic representative in the adiabatic limit of the image of \mathcal{S} in real cohomology. As a consequence of Theorem 5.3.2, this is of the form (Theorem 6.4.4)

$$[\mathcal{S}]_0 = \lim_{\delta \rightarrow 0} [\mathcal{S}]_{g_\delta} = \alpha(\Theta) - \pi^*H_{g, \Theta, \mathcal{S}} \in \Omega^3(P).$$

The choice of metric, connection, and string structure give a *canonical* 3-form $H_{g, \Theta, \mathcal{S}}$.

Letting \mathcal{A}_P denote the affine space of connections on P , the map

$$Met(M) \times \mathcal{A}_P \times \{\text{String Structures}\} \longrightarrow \Omega^3(M)$$

is equivariant with respect to the action of $H^3(M; \mathbb{Z})$, which acts on the space of string structures by adding the pulled back cohomology class, and which acts on $\Omega^3(M)$ by adding the harmonic representative of a class. In the above construction, $dH_{g,\Theta,s} = \frac{p_1}{2}(P, \Theta)$. Furthermore, $H_{g,\Theta,s}$ and the pulled back Chern–Simons form give the same values when evaluated on any 3-cycle in M . (The choice of string structure gives a global section, up to homotopy, on any 3-cycle in M .)

In particular, if $Spin(M) \rightarrow M$ is the frame bundle for a Riemannian manifold with spin structure, the metric induces the Levi-Civita connection. Therefore, we have a map

$$Met(M) \times \{\text{String Structures}\} \longrightarrow \Omega^3(M)$$

which is equivariant under the natural action of $H^3(M; \mathbb{Z})$. Analogous statements also hold for \tilde{U} and $spin^c$ structures (Theorems 6.2.5 and 6.3.3).

In Chapter 7, we use this construction to define canonical metric connections associated to string structures. Given (M, g, \mathcal{S}) , we obtain the canonical 3-form $H_{g,\mathcal{S}}$ and define a metric connection $\nabla^{g,\mathcal{S}}$ on TM whose torsion is given by

$$\langle T^{g,\mathcal{S}}(X, Y), Z \rangle_g = H_{g,\mathcal{S}}(X, Y, Z).$$

Though the form $H_{g,\mathcal{S}}$ is independent of a global rescaling on M , the resulting connection uses the metric g to define the torsion tensor, and hence it is natural to introduce the parameter $\epsilon > 0$. This gives a 1-parameter family of connections $\nabla^{\epsilon g, \mathcal{S}}$, which preserve the metric g and have torsion $\frac{1}{\epsilon} T^{g,\mathcal{S}}$. If ∇^g is the Levi-Civita

connection, then the Ricci tensor for $\nabla^{\epsilon g, \mathcal{S}}$ is given by

$$Ric^{\epsilon g, \mathcal{S}}(X, Y) = Ric^g(X, Y) + \frac{1}{2\epsilon} \sum_i \langle (\nabla_{e_i}^g T)(X, Y), e_i \rangle - \frac{1}{4\epsilon^2} \sum_i \langle T_{e_i} X, T_{e_i} Y \rangle.$$

CHAPTER 2

PRELIMINARIES

The main theorems in this thesis involve principal bundles P , Hodge theory, and Chern–Simons forms. In this chapter, we introduce these concepts and the properties which we use. In particular, since we later restrict to the subcomplex of G -invariant forms on a principal bundle, we set up the description of TP as a G -equivariant bundle. Along the way we introduce the notion of Lie algebra cohomology and are always careful to avoid confusions between left and right.

2.1 Lie groups and torsors

For simplicity, we always deal with G a compact Lie group, and use L_g to denote the isomorphism given by left multiplication of $g \in G$ and R_g for right multiplication by $g \in G$.

$$\begin{array}{ll} L_g : G \rightarrow G & R_g : G \rightarrow G \\ g_1 \mapsto g \cdot g_1 & g_1 \mapsto g_1 \cdot g \end{array}$$

These induce maps on the tangent spaces

$$(L_g)_* : T_{g_1}G \rightarrow T_{g \cdot g_1}G, \quad (R_g)_* : T_{g_1}G \rightarrow T_{g_1 \cdot g}G,$$

and

$$(Ad_g)_* \stackrel{\text{def}}{=} (L_g)_* \circ (R_{g^{-1}})_* : T_{g_1}G \rightarrow T_{g \cdot g_1 \cdot g^{-1}}G.$$

We denote the induced map on the tangent bundles by

$$(L_g)_* : TG \rightarrow TG, \quad (R_g)_* : TG \rightarrow TG,$$

and the induced map on smooth vector fields

$$(L_g)_* : C^\infty(G, TG) \rightarrow C^\infty(G, TG), \quad (R_g)_* : C^\infty(G, TG) \rightarrow C^\infty(G, TG).$$

Definition 2.1.1. *To any Lie group G we canonically associate the Lie algebra \mathfrak{g} of left-invariant vector fields:*

$$\mathfrak{g} \stackrel{\text{def}}{=} \{\text{left-invariant vector fields on } G\} = \{V \in C^\infty(G, TG) \mid (L_g)_* V = V \forall g \in G\}.$$

In other words, an element $V \in \mathfrak{g}$ is a map $V : G \rightarrow TG$ such that $V(g_1) \in T_{g_1}G$ for every g_1 , and

$$(L_g)_* V(g_1) = V(g \cdot g_1)$$

for every $g \in G$. The bracket $[\cdot, \cdot]$ is defined by the usual Lie bracket on vector fields.

Definition 2.1.2. *The Adjoint action of G on \mathfrak{g} is defined by*

$$Ad_g : \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$V \mapsto (L_g)_*(R_{g^{-1}})_* V = (R_{g^{-1}})_* V.$$

Given any vector in the tangent space of a point, we can associate to it a left-invariant vector field by left-translating the vector; i.e.

$$(L_G)_* : T_{g_1}G \rightarrow \mathfrak{g} \subset C^\infty(G, TG)$$

$$v \mapsto ((L_g)_{g_1} \mapsto (L_g)_* v).$$

Definition 2.1.3. *The Maurer-Cartan 1-form $\theta \in \Omega^1(G; \mathfrak{g})$ (often denoted $g^{-1}dg$), is the \mathfrak{g} -valued 1-form on G defined by associating to a tangent vector the vector field obtained by left-multiplication; i.e.*

$$\begin{aligned}\theta : TG &\rightarrow \mathfrak{g} \\ (g, v) &\mapsto (L_G)_*v.\end{aligned}$$

Proposition 2.1.4. *The Maurer-Cartan 1-form $\theta \in \Omega^1(G; \mathfrak{g})$ is left-invariant, and is right-equivariant:*

$$L_g^*\theta = \theta, \quad R_g^*\theta = Ad_{g^{-1}}\theta.$$

It also satisfies the equation

$$d\theta = -\frac{1}{2}[\theta \wedge \theta].$$

Proof. For any vector field $V \in \mathfrak{g}$, $(L_g)_*V = V$ and $(R_g)_*V = Ad_{g^{-1}}V$. For an arbitrary vector $v \in T_gG$, by definition $\theta(v) = (L_G)_*v \in \mathfrak{g}$. Therefore,

$$L_g^*\theta(v) = \theta((L_g)_*v) = (L_G)_*((L_g)_*v) = (L_G)_*v = \theta(v),$$

and

$$R_g^*\theta(v) = \theta((R_g)_*v) = (L_G)_*((R_g)_*v) = R_{g*}(L_G)_*v = Ad_{g^{-1}}\theta(v).$$

To evaluate the derivative, it suffices to use left-invariant vector fields V_0, V_1 on G . Then, as a function with values in a fixed vector space \mathfrak{g} , $\theta(V_i)$ is constant. Hence,

$$\begin{aligned}d\theta(V_0, V_1) &= V_0\theta(V_1) - V_1\theta(V_0) - \theta([V_0, V_1]) \\ &= -\theta([V_0, V_1]) = -\frac{1}{2}([\theta(V_0), \theta(V_1)] - [\theta(V_1), \theta(V_0)]) \\ &= -\frac{1}{2}[\theta \wedge \theta](V_0, V_1).\end{aligned}$$

□

Now, we wish to discuss right G -torsors. A torsor for a group G is a space Y on which G acts freely and transitively. Torsors appear frequently, and in our case the fibers of a principal bundle are torsors.

Suppose Y is a smooth manifold that has a smooth right action of G which is free and transitive

$$Y \times G \rightarrow Y.$$

Choosing a point $y_0 \in Y$ gives us the diffeomorphism $\varphi_{y_0} : Y \rightarrow G$ by

$$Y \xrightarrow{\varphi_{y_0}} G$$

$$y = y_0 \cdot g \longmapsto g.$$

Note that φ_{y_0} is equivariant with respect to right multiplication by G . Choosing a different point $y_1 \in Y$ gives another diffeomorphism which differs by *left* multiplication in G . In other words, since $y_0 = y_1 \cdot g_{y_1 y_0}$, the following diagram commutes:

$$\begin{array}{ccc}
 & & g \\
 & \nearrow \varphi_{y_0} & \downarrow L_{g_{y_1 y_0}} \\
 y_0 \cdot g = y_1 \cdot g_{y_1 y_0} \cdot g & & g_{y_1 y_0} \cdot g \\
 & \searrow \varphi_{y_1} & \\
 & &
 \end{array}
 \tag{2.1.1}$$

Therefore, $\varphi_{y_1} \circ \varphi_{y_0}^{-1} = L_{g_{y_1 y_0}} : G \rightarrow G$. This implies that any structure on G which is invariant under left multiplication induces a structure on the torsor Y . In particular, the Maurer-Cartan 1-form on $\theta \in \Omega^1(G; \mathfrak{g})$ pulls back to a Lie-algebra valued 1-form on the torsor Y . We will also call this form $\theta \in \Omega^1(Y; \mathfrak{g})$.

Proposition 2.1.5. *If Y is a right G -torsor, there is a canonical $\theta = \varphi_y^* \theta \in \Omega^1(Y; \mathfrak{g})$. This induces the canonical G -equivariant isomorphism*

$$TY \cong Y \times \mathfrak{g}.$$

Proof. Choosing some point $y_0 \in Y$ gives $\varphi_{y_0}^* \theta \in \Omega^1(Y; \mathfrak{g})$. A choice of a different point y_1 differs only by left multiplication in G . By the left-invariance of θ (Proposition 2.1.4) and the fact that φ_{y_0} and φ_{y_1} differ by left-multiplication (2.1.1),

$$\varphi_{y_1}^* \theta = \varphi_{y_0}^* L_{g_{y_1 y_0}}^* \theta = \varphi_{y_0}^* \theta.$$

Therefore, we have a well-defined \mathfrak{g} -valued 1-form on Y which we will also denote θ . At each tangent space, we then see the canonical vector space isomorphism

$$\begin{aligned} \theta : T_y Y &\xrightarrow{\cong} \mathfrak{g} \\ v &\longmapsto \theta(v) \end{aligned}$$

and therefore the isomorphism

$$TY \xrightarrow{Y \times \theta} Y \times \mathfrak{g}.$$

By Proposition 2.1.4, $\theta((R_g)_* v) = Ad_{g^{-1}} \theta(v)$. Hence, this isomorphism is equivariant with respect to the right G action on TY and \mathfrak{g} . \square

We now define the abstract vector space

$$\mathfrak{g}_Y \stackrel{\text{def}}{=} Y \times_{Ad} \mathfrak{g} = (Y \times \mathfrak{g}) / ((y, v) \sim (yg, Ad_{g^{-1}} v)). \quad (2.1.2)$$

It is *non-canonically* isomorphic to \mathfrak{g} . Once we choose a point $y \in Y$, then we have an isomorphism given by the fact that any element of \mathfrak{g}_Y can be written as (y, v) for $v \in \mathfrak{g}$. If we consider this vector space as a vector bundle over a point

$$\mathfrak{g}_Y \longrightarrow pt,$$

and we have the map $Y \xrightarrow{\pi} pt$, then we have the following equivalence of bundles.

Proposition 2.1.6. *There are canonical isomorphisms of G -equivariant vector bun-*

dles

$$TY \cong Y \times \mathfrak{g} \cong \pi^* \mathfrak{g}_Y.$$

Proof. The first isomorphism was given in Proposition 2.1.5. To see the second, note that by definition,

$$\pi^* \mathfrak{g}_{Y|y} = \{(y, e) \mid e \in \mathfrak{g}_Y\}.$$

Using the chosen y , we have the canonical isomorphism $\mathfrak{g}_Y \cong \mathfrak{g}$. Therefore,

$$\pi^* \mathfrak{g}_Y \cong Y \times \mathfrak{g}.$$

□

2.2 Right invariant vector fields and forms on a G -torsor

What do the right-invariant vector fields on a torsor, denoted $C^\infty(Y, TY)^G$, look like? Using the isomorphism in Proposition 2.1.6, we have the following equivalence:

$$C^\infty(Y, TY)^G \cong C^\infty(Y, Y \times \mathfrak{g})^G \cong C^\infty(Y, \pi^* \mathfrak{g}_Y)^G \cong \pi^* C^\infty(pt, \mathfrak{g}_Y).$$

More concretely, we have natural equivalences between right-invariant vector fields, elements of \mathfrak{g}_Y , and functions of the form

$$\tilde{V} : Y \rightarrow \mathfrak{g} \text{ such that } \tilde{V}(yg) = Ad_{g^{-1}} \tilde{V}(y).$$

When using this translation, for a given vector $V \in \mathfrak{g}_Y$ we will denote the corresponding vector field by $\tilde{V} : Y \rightarrow \mathfrak{g}$. Because the Lie bracket $[\cdot, \cdot]$ on \mathfrak{g} is Ad equivariant (or $G \xrightarrow{Ad} Gl(\mathfrak{g})$ is a representation), the induced bracket

$$\begin{aligned} (Y \times \mathfrak{g}) \times (Y \times \mathfrak{g}) &\xrightarrow{[\cdot, \cdot]} (Y \times \mathfrak{g}) \\ (p, V_1) \times (p, V_2) &\longmapsto (p, [V_1, V_2]) \end{aligned}$$

is Ad-equivariant and descends to

$$\mathfrak{g}_Y \times \mathfrak{g}_Y \xrightarrow{[\cdot, \cdot]} \mathfrak{g}_Y. \quad (2.2.1)$$

More concretely, the bracket is defined by

$$\widetilde{[V_1, V_2]}(y) = [\widetilde{V}_1(y), \widetilde{V}_2(y)]. \quad (2.2.2)$$

Lemma 2.2.1. *The Lie bracket of right-invariant vector fields on Y is given by the bracket on \mathfrak{g}_Y from (2.2.1); i.e. the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{g}_Y \times \mathfrak{g}_Y & \xrightarrow{[\cdot, \cdot]} & \mathfrak{g}_Y \\ \downarrow \approx & & \downarrow \approx \\ C^\infty(Y, TY)^G \times C^\infty(Y, TY)^G & \xrightarrow{[\cdot, \cdot]} & C^\infty(Y, TY)^G. \end{array}$$

Proof. Let $V_1, V_2 \in \mathfrak{g}_Y$, and $\widetilde{V}_1, \widetilde{V}_2 : Y \rightarrow \mathfrak{g}$ the induced vector fields in $C^\infty(Y, TY)^G$. Because the \widetilde{V}_i are right-invariant, around any point $y \in Y$ we have that $\widetilde{V}_i(yg) = Ad_{g^{-1}}\widetilde{V}_i(y)$. Therefore, the bracket $[\widetilde{V}_1, \widetilde{V}_2](y) \in \mathfrak{g}$ is given by

$$[\widetilde{V}_1, \widetilde{V}_2](y) = [Ad_{g^{-1}}\widetilde{V}_1(y), Ad_{g^{-1}}\widetilde{V}_2(y)](y) = \left(Ad_{g^{-1}}[\widetilde{V}_1(y), \widetilde{V}_2(y)] \right) (y) = [\widetilde{V}_1(y), \widetilde{V}_2(y)].$$

The middle equality follows from the Ad-equivariance of the Lie bracket. By definition of the bracket on \mathfrak{g}_Y (2.2.2), the above diagram is commutative. \square

To discuss right-invariant differential forms, we define the vector spaces

$$\Lambda^k \mathfrak{g}_Y^* \stackrel{\text{def}}{=} Y \times_{Ad} \Lambda^k \mathfrak{g}^* = Y \times \Lambda^k \mathfrak{g}^* / ((y, \psi) \sim (yg, Ad_{g^{-1}}^* \psi)).$$

Then, we have canonical isomorphisms

$$\Omega^k(Y)^G \cong C^\infty(Y, Y \times \Lambda^k \mathfrak{g}^*)^G = \pi^* C^\infty(pt, \Lambda^k \mathfrak{g}_Y^*). \quad (2.2.3)$$

This gives us an isomorphism of cochain complexes

$$\{\Omega^k(Y)^G, d\} \cong \{\Lambda^k \mathfrak{g}_Y^*, d_{\mathfrak{g}}\},$$

where $d_{\mathfrak{g}}$ is at this point only defined by d and the isomorphism (2.2.3). We now wish to give a more intrinsic definition of $d_{\mathfrak{g}}$. Let $\psi \in \Lambda^k \mathfrak{g}_Y^*$, $V_0, \dots, V_k \in \mathfrak{g}_Y$, and $\tilde{\psi} = \pi^* \psi$, $\tilde{V}_i = \pi^* V_i$. Then,

$$\begin{aligned} d_{\mathfrak{g}} \psi(V_0, \dots, V_k) &= d\tilde{\psi}(\tilde{V}_0, \dots, \tilde{V}_k) \\ &= \sum_i (-1)^i \tilde{V}_i \tilde{\psi}(\tilde{V}_0, \dots, \widehat{\tilde{V}_i}, \dots, \tilde{V}_k) \\ &\quad + \sum_{i < j} (-1)^{i+j} \tilde{\psi}([\tilde{V}_i, \tilde{V}_j], \dots, \widehat{\tilde{V}_i}, \dots, \widehat{\tilde{V}_j}, \dots, \tilde{V}_k). \end{aligned}$$

Because $\tilde{\psi}$ and \tilde{V}_i are right-invariant, then $\tilde{\psi}(\tilde{V}_0, \dots, \widehat{\tilde{V}_i}, \dots, \tilde{V}_k)$ is a constant function on Y , and its derivative is 0 along any vector field. Therefore,

$$d_{\mathfrak{g}} \psi(V_0, \dots, V_k) = \sum_{i < j} (-1)^{i+j} \psi([V_i, V_j], \dots, \widehat{V_i}, \dots, \widehat{V_j}, \dots, V_k).$$

This gives an intrinsic definition of $d_{\mathfrak{g}}$ in terms of the Lie bracket that coincides with the exterior derivative of right-invariant forms; i.e. the following diagram is commutative:

$$\begin{array}{ccc} \Lambda^k \mathfrak{g}_Y^* & \xrightarrow{d_{\mathfrak{g}}} & \Lambda^{k+1} \mathfrak{g}_Y^* \\ \downarrow \pi^* & & \downarrow \pi^* \\ \Omega^k(Y)^G & \xrightarrow{d} & \Omega^{k+1}(Y)^G. \end{array} \quad (2.2.4)$$

A classical theorem of Chevalley and Eilenberg shows that the inclusion of cochain complexes

$$\{\text{right-invariant forms on } Y\} \hookrightarrow \Omega^*(Y)$$

induces an isomorphism on the cohomology of these complexes (see Theorem 2.3 in

[CE]). Therefore, to compute the de Rham cohomology of Y , it suffices to look at the finite dimensional complex of right-invariant forms, which is much easier than looking at the infinite dimensional complex of arbitrary forms. As seen above, it is easy to explicitly describe the derivative $d_{\mathfrak{g}}$ via information encoded in the Lie algebra \mathfrak{g} . In fact, the first and last $d_{\mathfrak{g}}$ are both trivial.

Lemma 2.2.2. *For Y a right G torsor, the derivatives $d_{\mathfrak{g}} : \mathbb{R} \rightarrow \mathfrak{g}_Y^*$ and $d_{\mathfrak{g}} : \Lambda^{n-1}\mathfrak{g}_Y^* \rightarrow \Lambda^n\mathfrak{g}_Y^*$ are both 0 (where $n = \dim \mathfrak{g}$).*

Proof. Note that $H^0(\mathfrak{g})$ and $H^n(\mathfrak{g})$ are both isomorphic to \mathbb{R} (G is compact) and

$$\dim \Lambda^0 \mathfrak{g}_Y^* = \dim \Lambda^n \mathfrak{g}_Y^* = 1.$$

Therefore, $\text{Ker}(d_{\mathfrak{g}} : \Lambda^0 \mathfrak{g}_Y^* \rightarrow \Lambda^1 \mathfrak{g}_Y^*) = \Lambda^0 \mathfrak{g}_Y^*$, and $\text{Image}(d_{\mathfrak{g}} : \Lambda^{n-1} \mathfrak{g}_Y^* \rightarrow \Lambda^n \mathfrak{g}_Y^*) = 0$.

□

Therefore, the cochain complex of right-invariant forms is canonically isomorphic to

$$0 \rightarrow \mathbb{R} \xrightarrow{0} \mathfrak{g}_Y^* \xrightarrow{d_{\mathfrak{g}}} \Lambda^2 \mathfrak{g}_Y^* \xrightarrow{d_{\mathfrak{g}}} \dots \xrightarrow{d_{\mathfrak{g}}} \Lambda^{n-1} \mathfrak{g}_Y^* \xrightarrow{0} \Lambda^n \mathfrak{g}_Y^* \rightarrow 0.$$

2.3 Connections on principal G -bundles

Again, we assume G to be a compact Lie group. Informally, a principal G -bundle P is a *family* of right G torsors parameterized by a space M . G acts (from the right) freely and transitively on each fiber. The previous constructions on right G -torsors were canonical and will induce the same constructions on a principal bundle. See section 1 of [Fre1] for another brief overview of connections on principal bundles. Details and explicit proofs are found in sections II.2, II.3, II.5, and III.1 of [KN].

Definition 2.3.1. *A principal G -bundle $P \xrightarrow{\pi} M$ is a manifold P with a free right G action. Then $M = P/G$ is also a manifold.*

Example 2.3.2. *The usual example of a principal G -bundle is the frame bundle for some vector bundle. Assume that $G \subset M_n(\mathbb{F})$ (where $\mathbb{F} = \mathbb{R}$ or \mathbb{C}) is defined as the group of linear transformations which preserve some structure on \mathbb{F}^n . For example, $O(n)$ is the group of linear transformations which preserves the standard metric on \mathbb{R}^n , $SO(n)$ preserves the metric and orientation, and $U(n)$ preserves the standard hermitian metric on \mathbb{C}^n . Then, given a vector bundle $E \rightarrow M$ equipped with said structure, define the set of structure preserving frames on E_x*

$$G(E_x) = \{\mathbb{F}^n \xrightarrow{f} E_x \mid f \text{ is compatible with structure}\}.$$

$G(E_x)$ is a right torsor for G , where the action is given by precomposing with $g : \mathbb{F}^n \rightarrow \mathbb{F}^n$. The spaces $G(E_x)$ are parameterized by M , and taken together form a principal bundle $G(E) \xrightarrow{\pi} M$.

Let $P_x \stackrel{\text{def}}{=} \pi^{-1}(x)$ denote the fiber at a point $x \in M$. Because G acts freely on P , then P_x has a free and transitive right G action. In other words, P_x is a right G -torsor. Define the distribution of vertical tangent vectors by

$$T^V P \stackrel{\text{def}}{=} \text{Ker}(\pi_*) \subset TP.$$

At any point $p \in P$, where $\pi(p) = x$,

$$T^V P|_p = \text{Ker}(\pi_*)|_p = T(\text{Ker } \pi)|_p = (TP_x)|_p.$$

Since P_x is a G -torsor, then the Maurer-Cartan 1-form $\theta \in \Omega^1(P_x, \mathfrak{g})$ gives the canonical G -equivariant isomorphism (Proposition 2.1.5)

$$TP_x \cong P_x \times \mathfrak{g}$$

$$(p, v) \mapsto (p, \theta(v)).$$

We define the adjoint bundle $\mathfrak{g}_P \rightarrow M$ as

$$(\mathfrak{g}_P \rightarrow M) \stackrel{\text{def}}{=} P \times_{Ad} \mathfrak{g} = (P \times \mathfrak{g}) / ((pg, Ad_{g^{-1}}v) (p, v)).$$

This is simply a parameterized version of the vector space in equation (2.1.2). Since the isomorphisms in Proposition 2.1.6 were canonical, we then have the isomorphisms of G -equivariant vector bundles

$$T^V P \cong P \times \mathfrak{g} \cong \pi^* \mathfrak{g}_P. \quad (2.3.1)$$

Though there is a canonical vertical subspace at each point, one must make a choice in picking out a horizontal subspace. This is due to the that the fibers of P are not canonically isomorphic to G , and so there is no intrinsic way to identify separate fibers with each other. This problem is resolved by introducing a connection on P , which is really just an equivariant projection from TP onto the vertical tangent bundle $T^V P$.

Definition 2.3.3. *A connection on $P \rightarrow M$ is a \mathfrak{g} -valued 1-form $\Theta \in \Omega^1(P; \mathfrak{g})$ such that*

1. $i_x^* \Theta = \theta \in \Omega^1(P_x; \mathfrak{g})$,
2. $R_g^* \Theta = Ad_{g^{-1}} \Theta$,

where $i_x : P_x \hookrightarrow P$ is the fiber-wise inclusion.

The first condition is equivalent to $\Theta : T_p P \rightarrow \mathfrak{g} \cong T_p^V P$ being the identity map on vertical vectors. The second condition implies that the projections are equivariant, since $\Theta(R_{g*}v) = Ad_{g^{-1}}\Theta(v)$ for any vector.

Although the vertical subspaces in P are canonically defined, it is the connection $\Theta \in \Omega^1(P; \mathfrak{g})$ that picks out a horizontal subspace at each point $p \in P$. This gives a right-equivariant distribution

$$T^H P \stackrel{\text{def}}{=} \text{Ker}(\Theta) \subset TP, \quad (2.3.2)$$

and $T^H P \oplus T^V P = TP$. A connection Θ then gives the canonical isomorphism

$$\pi_* : T_p^H P \xrightarrow{\cong} T_{\pi(x)} M. \quad (2.3.3)$$

This induces the canonical isomorphism at every point $p \in P$,

$$\pi^* TM_p = \{v \in TM_{\pi(p)}\} \xrightarrow{\cong} \{v \in T^H P_p\}.$$

and hence the canonical isomorphism of G -equivariant vector bundles

$$T^H P \cong \pi^* TM. \quad (2.3.4)$$

A connection also gives a way to identify separate fibers by parallel translation.

First choose a path

$$\gamma : [0, 1] \rightarrow M,$$

but we only consider $[0, 1]$ as a smooth manifold. We have not defined a metric on $[0, 1]$ (but use the numbers for notational convenience). Then $d\gamma : T[0, 1] \rightarrow TM$ gives a 1-dimensional distribution in TM , which we refer to as $d\gamma$. Using the isomorphism induced by the connection in (2.3.3), the distribution $d\gamma$ lifts to a 1-dimensional distribution $\widetilde{d\gamma} \subset TP_{|\pi^{-1}(\gamma)}$. Any 1-dimensional distribution is integrable, so for any point $p_0 \in P_{\gamma(0)}$ there exists a unique integral submanifold $\widetilde{\gamma}_{p_0} : [0, 1] \rightarrow P$ that is everywhere tangent to the distribution and $\widetilde{\gamma}_{p_0}(0) = p_0$. The horizontal distribution $T^H P$ is equivariant, and hence we have a G -equivariant map called *parallel translation*

$$\begin{aligned} \parallel_\gamma : P_{\gamma(0)} &\longrightarrow P_{\gamma(1)} \\ p_0 &\longmapsto \widetilde{\gamma}_{p_0}(1). \end{aligned}$$

This construction does not involve a metric on $[0, 1]$ and therefore is invariant under diffeomorphisms of $[0, 1]$, or reparameterizations of the path γ .

The horizontal subspace also induces covariant differentiation in associated vector bundles. Let W be a G representation, denoted $\rho : G \rightarrow Gl(W)$. The associated vector bundle is defined by

$$W_P = P \times_G W = P \times W / ((p, w) (pg, \rho(g^{-1})w))$$

and sections $C^\infty(M, W_P)$ are equivalent to equivariant functions $P \rightarrow W$. In order to take the derivative of a function, its values must live in a fixed vector space, and sections of a vector bundle live in a parameterized family of vector spaces. However, the values of functions $P \rightarrow W$ live in a fixed vector space. Let $\xi \in C^\infty(M, W_P)$ be a section of the vector bundle and $\tilde{\xi} \in C^\infty(P, W)$ the associated equivariant function. Given a vector field $X \in C^\infty(M, TM)$, there is a unique horizontal vector field $\tilde{X} \in C^\infty(P, T^H P)$ such that $\pi_* \tilde{X} = X$. It then makes sense to derivate ξ along the vector field \tilde{X}

$$d_{\tilde{X}} \tilde{\xi} \in C^\infty(P, W).$$

Because the function \tilde{f} and the vector field \tilde{X} are G -equivariant, $d_{\tilde{X}} \tilde{\xi}$ is also G -equivariant. Define $\nabla_X \xi \in C^\infty(M, P_W)$ to be unique section such that

$$\widetilde{\nabla_X \xi} = d_{\tilde{X}} \tilde{\xi}. \tag{2.3.5}$$

The operator $\nabla : C^\infty(M, E_P) \rightarrow C^\infty(M, T^*M \otimes E_P)$ is called the connection on the associated vector bundle. We also denote this

$$\Omega^0(M; W_P) \xrightarrow{\nabla} \Omega^1(M; W_P).$$

It is important to remember that ∇ depends on the choice of a connection on P in order to lift the vector field on M to a vector field on P . For $f \in C^\infty(M)$, ∇ also satisfies the Leibniz rule

$$\nabla(f\xi) = (df)\xi + f\nabla\xi.$$

This is seen by lifting $f\xi$ to $\widetilde{f\xi}$ and derivating. We can then extend ∇ as first order differential operator to the complex

$$\Omega^0(M; W_P) \xrightarrow{\nabla} \Omega^1(M; W_P) \xrightarrow{d_\nabla} \Omega^2(M; W_P) \xrightarrow{d_\nabla} \Omega^3(M; W_P) \xrightarrow{d_\nabla} \dots \quad (2.3.6)$$

by requiring that

$$d_\nabla(\omega \wedge \xi) = (d\omega) \wedge \xi + (-1)^i \omega \wedge (\nabla \xi)$$

for $\omega \in \Omega^i(M)$ and $\xi \in C^\infty(M, W_P)$.

The possibility that $T^H P$ is not an integrable distribution leads to the notion of curvature.

Definition 2.3.4. *Given a connection $\Theta \in \Omega^1(P; \mathfrak{g})$, the curvature 2-form $\Omega \in \Omega^2(P; \mathfrak{g})$ is*

$$\Omega \stackrel{\text{def}}{=} d\Theta + \frac{1}{2}[\Theta \wedge \Theta].$$

Proposition 2.3.5. *$R_g^* \Omega = Ad_{g^{-1}} \Omega$, and Ω is only non-zero when evaluated on horizontal vectors.*

Proof. G equivariance of Ω follows from the equivariance of Θ .

The Maurer-Cartan equation in Proposition 2.1.4 and the fact that $i_x^* \Theta = \theta$ for the inclusion $i_x : P_x \hookrightarrow P$ imply

$$i_x^* \Omega = d\theta + \frac{1}{2}[\theta \wedge \theta] = 0.$$

Therefore, Ω is zero when evaluated on two vertical vectors.

Let X_H be a horizontal vector and X_V a vertical vector. Then

$$\Omega(X_H, X_V) = X_H \Theta(X_V) - X_V \Theta(X_H) - \Theta([X_H, X_V]).$$

By definition, $\Theta(X_H) = 0$ and $\Theta(X_V) = X_V$. Viewing X_V as a function $P \rightarrow \mathfrak{g}$, we

see that

$$X_H\Theta(X_V) - \Theta([X_H, X_V]) = d_{X_H}X_V - d_{X_V}X_H = 0.$$

□

Since Ω is a G -equivariant, purely horizontal 2-form on P , we can view it as an element of $\Omega^2(M; \mathfrak{g}_P)$. From the definition, we see that given two horizontal vector fields V_{H_0}, V_{H_1} on P ,

$$\Omega(V_{H_0}, V_{H_1}) = d\Theta(V_{H_0}, V_{H_1}) = -\Theta([V_{H_0}, V_{H_1}]). \quad (2.3.7)$$

In other words, the curvature is (up to sign) given by taking the Lie bracket of two horizontal vector fields and then projecting onto the fiber. The curvature measures the non-integrability of the distribution $T^H P$. A connection is called flat if $\Omega = 0$, or equivalently, if the distribution $T^H P$ is integrable.

2.4 Hodge theory

de Rham cohomology allows us to view real cohomology in a more geometric way by interpreting cohomology classes in terms of differential forms. However, a given cohomology class only determines a closed differential form up to an additional exact form. The basic idea of Hodge theory is that the introduction of a metric canonically picks out a particular differential form by requiring that the form also be coclosed with respect to an adjoint differential.

Let (V, g) be an n -dimensional, oriented, Euclidean vector space. The Euclidean metric g induces a metric $\langle \cdot, \cdot \rangle$ on $\oplus_k \Lambda^k V$. This also gives an isomorphism $\Lambda^n V \cong \mathbb{R}$, and we denote ω the element which maps to 1. (When $V = T_x M^*$, then $\omega = \text{vol}_x$.) This gives rise to an isomorphism, called the Hodge star

$$* : \Lambda^k V \rightarrow \Lambda^{n-k} V,$$

where, for $\gamma \in \Lambda^k V$, $*\gamma$ is the unique element determined by

$$\gamma \wedge \beta = \langle *\gamma, \beta \rangle \omega$$

for all $\beta \in \Lambda^{n-k} V$. Useful properties include

- $*(*\gamma) = (-1)^{k(n-k)}\gamma$,
- $*1 = \omega$.

Let (M, g) be a compact oriented n -dimensional Riemannian manifold without boundary. By the above construction, for every $x \in M$ we obtain

$$* : \Lambda^k T_x M^* \rightarrow \Lambda^{n-k} T_x M^*,$$

and therefore we have an isomorphism of smooth differential forms

$$* : \Omega^k(M) \rightarrow \Omega^{n-k}(M).$$

Define $d^* : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ by

$$d^* \stackrel{\text{def}}{=} (-1)^{n(k+1)+1} * d *.$$

An integration by parts argument shows that d^* is the adjoint for d . Namely, if (\cdot, \cdot) is the inner product on differential forms induced by integrating the pointwise inner product, then

$$(d\gamma, \beta) = (\gamma, d^*\beta). \tag{2.4.1}$$

In other words, the metric picks out an adjoint d^* to d and gives

$$0 \begin{array}{c} \xleftarrow{d^*} \\ \xrightarrow{d} \end{array} \Omega^0(M) \begin{array}{c} \xleftarrow{d^*} \\ \xrightarrow{d} \end{array} \Omega^1(M) \begin{array}{c} \xleftarrow{d^*} \\ \xrightarrow{d} \end{array} \Omega^2(M) \begin{array}{c} \xleftarrow{d^*} \\ \xrightarrow{d} \end{array} \dots \begin{array}{c} \xleftarrow{d^*} \\ \xrightarrow{d} \end{array} \Omega^{n-1}(M) \begin{array}{c} \xleftarrow{d^*} \\ \xrightarrow{d} \end{array} \Omega^n(M) \xrightarrow{d} 0.$$

The Hodge star gives an isomorphism between the cochain complex $(\Omega^*(M), d)$ and

its dual complex $(\Omega^*(M), d^*)$, and therefore

$$H_k(\Omega^*(M), d^*) \cong H^{n-k}(\Omega^*(M), d). \quad (2.4.2)$$

Now, define the Hodge Laplacian

$$\Delta^k \stackrel{\text{def}}{=} (d + d^*)^2 = dd^* + d^*d : \Omega^k(M) \rightarrow \Omega^k(M).$$

In fact, $d + d^*$ is formally self-adjoint, and therefore $\text{Ker}(d + d^*)^2 = \text{Ker}(d + d^*)$.

Since d increases the degree of a form while d^* decreases the degree,

$$\text{Ker } \Delta^k = \text{Ker } d \cap \text{Ker } d^* \subset \Omega^k(M).$$

We denote $\mathcal{H}_g^k(M) = \text{Ker } \Delta^k \subset \Omega^k(M)$ to be the harmonic k -forms.

Theorem 2.4.1. (Hodge) *Let (M, g) be a compact, oriented, Riemannian manifold without boundary. Then there is an orthogonal decomposition*

$$\Omega^k(M) = d\Omega^{k-1}(M) \oplus d^*\Omega^{k+1}(M) \oplus \mathcal{H}_g^k(M).$$

Consequently, in each de Rham class $[\beta]$ there is a unique harmonic representative, and $\mathcal{H}_g^k(M) \cong H^k(M; \mathbb{R})$.

Corollary 2.4.2. *Any form $\omega \in d^*\Omega^{k+1}(M)$ is uniquely determined by $d\omega$. Any form $\omega \in d\Omega^{k-1}(M)$ is uniquely determined by $d^*\omega$.*

Proof. Since d^* is the adjoint to d , the corollary follows from the isomorphisms

$$d^*\Omega^{k+1} \begin{array}{c} \xleftarrow{d^*} \\ \xrightarrow{d} \end{array} d\Omega^k(M).$$

We treat one case more concretely. Suppose $\omega_1, \omega_2 \in d^*\Omega^{k+1}(M)$, and $d\omega_1 = d\omega_2$. Then, $\omega_1 - \omega_2$ is closed, and hence in $d\Omega^{k-1}(M) \oplus \mathcal{H}_g^k(M)$. But, $\omega_1 - \omega_2 \in d^*\Omega^{k+1}(M)$, and hence $\omega_1 = \omega_2$. □

It should be noted that while d and d^* are both locally defined, solving $\Delta\psi = 0$ is a global problem and in general very difficult. However, if a group G acts by isometries on M , then this problem is more tractable. First, note that because $*$ is defined solely in terms of the oriented metric, then $*$ commutes with isometries. Therefore, the Hodge star maps the subspace of G -invariant forms to itself, and Δ commutes with isometries. By the following proposition, when finding harmonic forms it suffices to look at the subcomplex of G -invariant forms.

Proposition 2.4.3. *Suppose that G is a connected Lie group which acts on (M, g) by isometries. If $\psi \in \mathcal{H}_g^k(M)$, then ψ is invariant under the G action; i.e. $\psi \in \Omega^k(M)^G$.*

Proof. Let $\gamma : [0, 1] \rightarrow G$ be a path connecting the identity e to any element $h \in G$. Then, since $e^*\psi = \psi$, and $\Delta((\gamma(t))^*\psi) = \gamma(t)^*\Delta\psi = 0$, we have a path in the space $\mathcal{H}_g^k(M)$. The forms $\gamma(t)^*\psi$ all represent the same cohomology class, so this path must be constant. \square

In particular, if G is a compact Lie group equipped with a bi-invariant metric, then any harmonic form must be bi-invariant. At the Lie algebra level, this implies that any harmonic form $\psi \in \Lambda^k \mathfrak{g}$ must be Ad -invariant. Conversely, if ψ is an Ad -invariant form, then it must be closed, since

$$\begin{aligned}
0 &= \left. \frac{d}{dt} \right|_{t=0} (Ad_{e^{tX_0}}^* \psi(X_1, \dots, X_k)) = \left. \frac{d}{dt} \right|_{t=0} \psi(Ad_{e^{tX_0}} X_1, \dots, Ad_{e^{tX_0}} X_k) \\
&= \left. \frac{d}{dt} \right|_{t=0} \psi(Ad_{e^{tX_0}} X_1, X_2, \dots, X_k) + \dots + \left. \frac{d}{dt} \right|_{t=0} \psi(X_1, X_2, \dots, Ad_{e^{tX_0}} X_k) \\
&= \psi([X_0, X_1], X_2, \dots, X_k) + \dots + \psi(X_0, \dots, [X_0, X_k]) \\
&= d\psi(X_0, X_1, \dots, X_k).
\end{aligned}$$

(The third equality follows from the linearity of ψ .) Furthermore, if the metric on \mathfrak{g}

is Ad -invariant, then the Hodge star of Ad -invariant forms will be Ad -invariant and hence closed.

Proposition 2.4.4. *For an Ad -invariant metric on \mathfrak{g} , The harmonic forms on the finite-dimensional complex*

$$0 \longrightarrow \mathbb{R} \xrightarrow{0} \mathfrak{g}^* \xrightarrow{d_{\mathfrak{g}}^*} \Lambda^2 \mathfrak{g}^* \xrightarrow{d_{\mathfrak{g}}^*} \dots \xrightarrow{d_{\mathfrak{g}}^*} \Lambda^{n-1} \mathfrak{g}^* \xrightarrow{0} \Lambda^n \mathfrak{g}^* \longrightarrow 0$$

are precisely the Ad -invariant forms. This also implies that for G a compact and connected Lie group with a bi-invariant metric, the harmonic forms are precisely the bi-invariant forms.

We later deal with rescalings on the metric, so we prove the following useful lemma.

Lemma 2.4.5. *If $g_{\delta} = \delta^2 g$, then $*_{g_{\delta}} = \delta^{2k-n} *_{g} : \Lambda^k V \rightarrow \Lambda^{n-k} V$.*

Proof. The metric $\langle \cdot, \cdot \rangle$ on $\Lambda^{n-k} V$ is induced by

$$\langle v_1 \wedge \dots \wedge v_{n-k}, w_1 \wedge \dots \wedge w_{n-k} \rangle_g = \text{Det}(g(v_a, w_b)),$$

where $(g(v_a, w_b))$ denotes an $(n-k) \times (n-k)$ matrix. Then $\langle \cdot, \cdot \rangle_{\delta^2 g} = \delta^{2(n-k)} \langle \cdot, \cdot \rangle_g$.

The isometry $(V, g) \rightarrow (V, g_{\delta})$ sends $v \mapsto \delta^{-1} v$, and therefore

$$\omega_{g_{\delta}} = \delta^{-n} \omega_g.$$

Thus,

$$\begin{aligned} \gamma \wedge \beta &= \langle *_{g_{\delta}} \gamma, \beta \rangle_{g_{\delta}} \omega_{g_{\delta}} = \delta^{n-2k} \langle *_{g_{\delta}} \gamma, \beta \rangle_g \omega_g \\ &= \langle *_{g} \gamma, \beta \rangle_g \omega_g, \end{aligned}$$

and so $*_{g_{\delta}} = \delta^{2k-n} *_{g}$. □

2.5 Chern–Weil and Chern–Simons Forms

In Chapter 5, we only look at the Chern–Simons 1- and 3-forms, so the following treatment will be very concrete. The theory of Chern–Weil and Chern–Simons forms is quite general, though, and most of the properties stated here generalize. The paper [CS] contains these details, and sections 1.2-1.4 of [Fre2] contain an excellent summary of Chern–Weil and Chern–Simons forms. Section 1 of [Fre1] also proves the important properties of the Chern–Simons 3-form. In general, a Chern–Weil form lives in $\Omega^{2k}(M)$, and when pulled back to P , it is the derivative of a Chern–Simons form in $\Omega^{2k-1}(P)$.

Definition 2.5.1. *Given a $U(n)$ -bundle $P \xrightarrow{\pi} M$ with connection Θ , the first Chern-form is*

$$c_1(P, \Theta) \stackrel{\text{def}}{=} \frac{i}{2\pi} \text{Tr}(\Omega) \in \Omega^2(M).$$

where Tr is the trace of the standard matrix representation of the Lie algebra $\mathfrak{u}(n)$.

The form $c_1(P, \Theta)$ is a Chern–Weil form. Of course, $\Omega \in \Omega^2(M; \mathfrak{g}_P)$ takes values in the Adjoint bundle. However, since Tr is an Ad -invariant linear map $\mathfrak{g} \rightarrow \mathbb{R}$, we get a well-defined 2-form on M with values in \mathbb{R} . In general, Chern–Weil forms are obtained from applying an Ad -invariant polynomial to the curvature 2-form.

Definition 2.5.2. *The Chern–Simons 1-form of a $U(n)$ -bundle with connection (P, Θ) over M is*

$$\frac{i}{2\pi} \text{Tr}(\Theta) \in \Omega^1(P).$$

Proposition 2.5.3. *$\frac{i}{2\pi} \text{Tr}(\Theta) \in \Omega^1(P)$ satisfies the following properties.*

1. $i_x^* \left(\frac{i}{2\pi} \text{Tr}(\Theta) \right) = \frac{i}{2\pi} \text{Tr}(\theta) \in \Omega^1(G)$ is bi-invariant.
2. $\left[\frac{i}{2\pi} \text{Tr}(\theta) \right] \in H^1(U(n); \mathbb{R})$ is the \mathbb{R} -image of the standard generator in $H^1(U(n); \mathbb{Z}) \cong \mathbb{Z}$.

3. $R_g^*(\frac{i}{2\pi}Tr(\Theta)) = \frac{i}{2\pi}Tr(\Theta)$; i.e. $\frac{i}{2\pi}Tr(\Theta)$ is right-invariant.

4. $d\frac{i}{2\pi}Tr(\Theta) = \frac{i}{2\pi}Tr(\Omega) = \pi^*c_1(P, \Theta)$.

Proof. By definition, $i_x^*\Theta = \theta$, so $i_x^*(\frac{i}{2\pi}Tr(\Theta)) = \frac{i}{2\pi}Tr(\theta)$. The Ad -invariance of trace implies the bi-invariance of $\frac{i}{2\pi}Tr(\theta)$. In particular, this means that $\frac{i}{2\pi}Tr(\theta)$ is closed and represents a de Rham class.

To see the second property, first note that the inclusion $U(1) \hookrightarrow U(n)$ induces an isomorphism on the first cohomology. $[\frac{i}{2\pi}Tr(\theta)]$ is the real class representing the \mathbb{R} -image of the standard generator for $H^1(U(1); \mathbb{Z})$, as seen by

$$\int_{U(1)} \frac{i}{2\pi}Tr(\theta) = 1.$$

The third property follows from the right-equivariance of Θ and the Ad -invariance of trace, since

$$R_g^*(\frac{i}{2\pi}Tr(\Theta)) = \frac{i}{2\pi}Tr(Ad_{g^{-1}}\Theta) = \frac{i}{2\pi}Tr(\Theta).$$

Finally,

$$d\frac{i}{2\pi}Tr(\Theta) = \frac{i}{2\pi}Tr(d\Theta) = \frac{i}{2\pi}Tr(d\Theta + \frac{1}{2}[\Theta \wedge \Theta]) = \frac{i}{2\pi}Tr(\Omega) = \pi^*c_1(P, \Theta).$$

The middle equality follows from $Tr([\Theta \wedge \Theta]) = 0$. □

We now deal with the situation where G is a compact simple group (the main groups of interest will be $Spin(n)$).

Definition 2.5.4. Given a principal G -bundle $P \xrightarrow{\pi} M$ with connection Θ , where G is simple, we have the 4-dimensional Chern–Weil form given by

$$\langle \Omega \wedge \Omega \rangle \in \Omega^4(M),$$

where $\langle \cdot, \cdot \rangle$ denotes a suitably normalized Ad -invariant metric on \mathfrak{g} . For $G =$

$Spin(n)$, this is one-half the first Pontryagin form and denoted

$$\frac{p_1}{2}(P, \Theta) = \frac{-1}{16\pi^2} Tr(\Omega \wedge \Omega);$$

for $G = SO(n)$, this is the first Pontryagin form and denoted

$$p_1(P, \Theta) = \frac{-1}{8\pi^2} Tr(\Omega \wedge \Omega);$$

and for $G = SU(n)$, this is the second Chern form and denoted

$$c_2(P, \Theta) = \frac{1}{8\pi^2} Tr(\Omega \wedge \Omega),$$

where Tr always denotes the trace of the standard matrix representation.

Definition 2.5.5. For G simple, the Chern–Simons 3-form of a connection Θ on a principal G -bundle P is

$$\alpha(\Theta) \stackrel{def}{=} \langle \Theta \wedge \Omega \rangle - \frac{1}{6} \langle \Theta \wedge [\Theta \wedge \Theta] \rangle \in \Omega^3(P).$$

Proposition 2.5.6. The Chern–Simons 3-form has the following properties.

1. $i_x^* \alpha(\Theta) = -\frac{1}{6} \langle \theta \wedge [\theta \wedge \theta] \rangle \in \Omega^3(G)$ is bi-invariant.
2. $\frac{-1}{6} \langle \theta \wedge [\theta \wedge \theta] \rangle \in H^3(G; \mathbb{R})$ is the real cohomology class associated to the \mathbb{R} -image of the (standard) generator of $H^3(G; \mathbb{Z}) \cong \mathbb{Z}$ (in the stable range of G).
3. $R_g^* \alpha(\Theta) = \alpha(\Theta)$; i.e. $\alpha(\Theta)$ is right-invariant.
4. $d\alpha(\Theta) = \langle \Omega \wedge \Omega \rangle \in \Omega^4(P)$.

Proof. The first statement follows by $i_x^* \Theta = \theta$, along with the Ad -invariance of the Lie bracket and inner product $\langle \cdot, \cdot \rangle$.

Consequently, $-\frac{1}{6}\langle\theta \wedge [\theta \wedge \theta]\rangle$ is closed and represents a de Rham cohomology class. The form $\langle\theta \wedge [\theta \wedge \theta]\rangle \in \Lambda^3\mathfrak{g}^*$ is, up to constant, the standard generator for $H^3(\mathfrak{g})$ of any simple Lie algebra. This can be seen by evaluating it on a $su(2)$ subalgebra. The normalization factor in the definition of $\langle\cdot, \cdot\rangle$ serves to make it the \mathbb{R} -image of a generator of $H^3(G; \mathbb{Z})$ (at least in the stable range; otherwise see the discussion below). The factor itself is unimportant for our purposes, so we merely define it such that it satisfies this property.

The right-invariance of $\alpha(\Theta)$ follows from the right-equivariance of Θ and Ω , along with the Ad -invariance of trace, since

$$\begin{aligned} R_g^*\alpha(\Theta) &= \langle Ad_{g^{-1}}\Theta \wedge Ad_{g^{-1}}\Omega \rangle - \frac{1}{6}\langle Ad_{g^{-1}}\Theta \wedge [Ad_{g^{-1}}\Theta \wedge Ad_{g^{-1}}\Theta] \rangle \\ &= \langle \Theta \wedge \Omega \rangle - \frac{1}{6}\langle \Theta \wedge [\Theta \wedge \Theta] \rangle = \alpha(\Theta). \end{aligned}$$

Finally, the derivative of $\alpha(\Theta)$ is given by the calculation

$$\begin{aligned} d\alpha(\Theta) &= d\langle\Theta \wedge \Omega\rangle - d\frac{1}{6}\langle\Theta \wedge [\Theta \wedge \Theta]\rangle \\ &= \langle d\Theta \wedge \Omega \rangle - \langle \Theta \wedge d\Omega \rangle - \frac{1}{2}\langle d\Theta \wedge [\Theta \wedge \Theta] \rangle \\ &= \langle (\Omega - \frac{1}{2}[\Theta \wedge \Theta]) \wedge \Omega \rangle + \langle \Theta \wedge [\Theta \wedge \Omega] \rangle - \frac{1}{2}\langle (\Omega - \frac{1}{2}[\Theta \wedge \Theta]) \wedge [\Theta \wedge \Theta] \rangle \\ &= \langle \Omega \wedge \Omega \rangle - \frac{1}{4}\langle \Theta \wedge [\Theta \wedge [\Theta \wedge \Theta]] \rangle \\ &= \langle \Omega \wedge \Omega \rangle. \end{aligned}$$

The above equalities follow from the definition of curvature, a standard Bianchi identity

$$d\Omega + [\Theta \wedge \Omega] = 0,$$

the fact that $\langle\psi_1 \wedge [\psi_2 \wedge \psi_3]\rangle = \langle\psi_2 \wedge [\psi_3 \wedge \psi_1]\rangle$, and the Jacobi identity. \square

We now make some remarks about the Chern–Weil and Chern–Simons forms for $G = Spin(n)$ with $n \leq 4$, since $Spin(1)$, $Spin(2)$, and $Spin(4)$ are not simple

groups. Definitions 2.5.4 and 2.5.5 both make sense for these groups, but we should be careful about the statement of the second property in Proposition 2.5.6. The characteristic class $\frac{p_1}{2}(P) \in H^4(M; \mathbb{Z})$ is a stable class. This means that is the pullback of

$$\frac{p_1}{2} \in H^4(BSpin; \mathbb{Z})$$

where $BSpin$ is the direct limit $\varinjlim BSpin(n)$. The 3-dimensional cohomology of $Spin(n)$ and the 4-dimensional cohomology of $BSpin(n)$ both stabilize at $n = 5$. Thus, the inclusions $Spin(5) \hookrightarrow Spin$ and $BSpin(5) \hookrightarrow BSpin$ induce the identity maps on 3- and 4-dimensional cohomology, respectively. Therefore, the statements above are clear for $Spin(n)$ ($n \geq 5$), since we have a standard generator of $H^3(Spin(n); \mathbb{Z}) \cong \mathbb{Z}$.

The correct statement for $Spin(n)$ ($n \leq 4$) is that the cohomology class $\frac{-1}{6}(\theta \wedge [\theta \wedge \theta]) \in H^3(Spin(n); \mathbb{R})$ is the \mathbb{R} -image of the generator of $H^3(Spin(5); \mathbb{Z})$ under the inclusion $Spin(n) \hookrightarrow Spin(5)$. One can show $Spin(4) \cong SU(2) \times SU(2)$ and $Spin(3) \cong SU(2)$. Up to sign choices in the following notation, the generator from $H^3(Spin(5); \mathbb{Z}) \cong \mathbb{Z}$ gets sent to $(1, 1) \in \mathbb{Z} \oplus \mathbb{Z} \cong H^3(Spin(4); \mathbb{Z})$ and $2 \in \mathbb{Z} \cong H^3(Spin(3); \mathbb{Z})$ under the respective inclusions.

CHAPTER 3

RIGHT-INVARIANT FORMS ON A PRINCIPAL BUNDLE

In this chapter, we give an explicit description of the complex and dual complex of right-invariant forms on P . This description will be important in the calculations of Chapter 5. To make sense of the dual complex, we construct a Riemannian metric on P which depends on a metric on M and a connection Θ . We also introduce the rescaled derivative d_δ , the rescaled coderivative d_δ^* , and the rescaled Laplacian L_{g_δ} , all of which are fundamental in the spectral sequence of Chapter 4 and the calculations of Chapter 5.

3.1 Isomorphic bi-graded cochain complex

Let G be a compact Lie group and $P \xrightarrow{\pi} M$ a principal G -bundle with connection $\Theta \in \Omega^1(P; \mathfrak{g})$. The connection picks out a horizontal subspace at each point in P , and hence

$$TP = T^H P \oplus T^V P,$$

where $T^H P = \text{Ker}(\Theta)$ denotes the horizontal distribution and $T^V P = \text{Ker}(\pi_*)$ denotes the vertical distribution. This induces a bi-grading on the space of differential forms

$$\Omega^k(P) = \bigoplus_{i+j=k} \Omega^{i,j}(P) = \bigoplus_{i+j=k} C^\infty(P; \Lambda^i T^H P^* \otimes \Lambda^j T^V P^*).$$

The bi-grading notation will always be (horizontal, vertical). In other words, $\Omega^{i,j}(P)$ is a form which takes values on i horizontal and j vertical vectors. Furthermore, the exterior derivative decomposes as (Lemma 3.1.1)

$$d = d^{0,1} + d^{1,0} + d^{2,-1},$$

where $d^{a,b} : \Omega^{i,j}(P) \rightarrow \Omega^{i+a,j+b}(P)$. The right action of G preserves the decomposition of TP (i.e. it maps $T^H P \rightarrow T^H P$ and $T^V P \rightarrow T^V P$). It then makes sense to talk about the space $\Omega^{i,j}(P)^G$ of right-invariant (i, j) forms. In this next section, we show there is a natural isomorphism of bi-graded cochain complexes

$$\{\Omega^i(M; \Lambda^j \mathfrak{g}_P^*), d\} \xrightarrow{\pi^*} \{\Omega^{i,j}(P)^G, d\}.$$

Furthermore, the bi-graded decomposition of d in $\{\Omega^i(M; \Lambda^j \mathfrak{g}_P^*), d\}$ has a nice description. The $(0, 1)$ component is given, up to sign, by the vector bundle homomorphism induced from the derivative of right-invariant vector fields on P_x

$$\Lambda^k \mathfrak{g}_P^* \xrightarrow{d_{\mathfrak{g}}} \Lambda^{k+1} \mathfrak{g}_P^*.$$

The principal bundle connection Θ on P induces a vector bundle connection ∇ on the associated bundles $\Lambda^j \mathfrak{g}_P^*$. The $(1, 0)$ component of d is given by the unique extension of ∇ to the cochain complex

$$\dots \xrightarrow{d_{\nabla}} \Omega^i(M; \Lambda^j \mathfrak{g}_P^*) \xrightarrow{d_{\nabla}} \Omega^{i+1}(M; \Lambda^j \mathfrak{g}_P^*) \xrightarrow{d_{\nabla}} \dots$$

Finally, the curvature is an element $\Omega \in \Omega^2(M; \mathfrak{g})$, and $d^{2,-1}$ is given, up to sign, by contracting along this vector-valued 2-form

$$\Omega^i(M; \Lambda^j \mathfrak{g}_P^*) \xrightarrow{\iota_{\Omega}} \Omega^{i+2}(M; \Lambda^{j-1} \mathfrak{g}_P^*).$$

The bi-graded complex isomorphic to $\{\Omega^{i,j}(P)^G, d\}$ is written

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & & \vdots \\
& \uparrow d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} & & \uparrow d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} \\
0 & \longrightarrow & \Omega^0(M; \Lambda^2 \mathfrak{g}_P^*) & \xrightarrow{\nabla} & \Omega^1(M; \Lambda^2 \mathfrak{g}_P^*) & \xrightarrow{d_{\nabla}} & \Omega^2(M; \Lambda^2 \mathfrak{g}_P^*) & \xrightarrow{d_{\nabla}} & \Omega^3(M; \Lambda^2 \mathfrak{g}_P^*) & \xrightarrow{d_{\nabla}} & \cdots \\
& \uparrow d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} & & \uparrow d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} \\
0 & \longrightarrow & \Omega^0(M; \mathfrak{g}_P^*) & \xrightarrow{\nabla} & \Omega^1(M; \mathfrak{g}_P^*) & \xrightarrow{d_{\nabla} \iota_{\Omega}} & \Omega^2(M; \mathfrak{g}_P^*) & \xrightarrow{d_{\nabla} -\iota_{\Omega}} & \Omega^3(M; \mathfrak{g}_P^*) & \xrightarrow{d_{\nabla} \iota_{\Omega}} & \cdots \\
& \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 \\
0 & \longrightarrow & \Omega^0(M; \mathbb{R}) & \xrightarrow{d_M} & \Omega^1(M; \mathbb{R}) & \xrightarrow{d_M \iota_{\Omega}} & \Omega^2(M; \mathbb{R}) & \xrightarrow{d_M -\iota_{\Omega}} & \Omega^3(M; \mathbb{R}) & \xrightarrow{d_M \iota_{\Omega}} & \cdots \\
& \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0
\end{array}$$

We now proceed in proving the above statements.

First, let us look at the general situation of the exterior derivative on a bi-graded complex induced from two distributions.

Lemma 3.1.1. *Let P be generic manifold with two linearly independent distributions $A, B \subset TP$ such that $A + B = TP$. This induces a bi-grading on $\Omega^*(P)$,*

$$\Omega^{i,j}(P) = C^\infty(P, \Lambda^i A^* \otimes \Lambda^j B^*);$$

the exterior derivative decomposes as

$$d = d^{-1,2} + d^{0,1} + d^{1,0} + d^{2,-1},$$

where $d^{a,b} : \Omega^{i,j}(P) \rightarrow \Omega^{i+a,j+b}(P)$. The component $d^{-1,2} = 0$ if and only if B is integrable, and $d^{2,-1} = 0$ if and only if A is integrable.

Proof. The bi-grading on forms is a simple consequence of

$$\Lambda^k TP^* = \Lambda^k(A \oplus B)^* = \Lambda^k(A^* \oplus B^*) = \bigoplus_{i+j=k} \Lambda^i A^* \otimes \Lambda^j B^*.$$

To prove the second part, it suffices to show for 1-forms (due to the Leibniz rule).

Since

$$d : \Omega^{0,1}(P) \oplus \Omega^{1,0}(P) \rightarrow \bigoplus_{i+j=2} \Omega^{i,j}(P),$$

we see that d has at most 4 parts. The $d^{-1,2}$ and $d^{2,-1}$ components are due to the possible non-integrability of B and A , respectively. To see this, look at the formula for $d\psi$ where V_0, V_1 are vector fields:

$$d\psi(V_0, V_1) = V_0\psi(V_1) - V_1\psi(V_0) - \psi([V_0, V_1]).$$

In particular, if $\psi \in \Omega^{1,0}$, and $V_0, V_1 \in C^\infty(P; B)$, then

$$d^{2,-1}\psi(V_0, V_1) = -\psi([V_0, V_1]). \quad (3.1.1)$$

If B is integrable, then for any such vector fields in B , $[V_0, V_1] \in C^\infty(M; B)$, and therefore $d^{2,-1}\psi = 0$. However, if B is not integrable, there exist vector fields V_0, V_1 such that $[V_0, V_1]|_A \neq 0$. Letting $\psi \in \Omega^{1,0}$ be a differential form which is non-zero on $[V_0, V_1]$, we see that $d^{2,-1}\psi \neq 0$. The same argument shows that $d^{2,-1} = 0$ if and only if A is integrable. \square

Because P is a fiber bundle, the vertical distribution $T^V P$ is integrable. This follows immediately from

$$\pi_*[V_0, V_1] = [\pi_*V_0, \pi_*V_1].$$

Therefore, $d = d^{0,1} + d^{1,0} + d^{2,-1}$. However, $T^H P$ does not have to be integrable, and in fact, integrability of $T^H P$ is equivalent to Θ being a flat connection.

We now wish to look at the sub-complex of right-invariant forms on P

$$\{\Omega^{i,j}(P)^G, d\} \hookrightarrow \{\Omega^{i,j}(P), d\}.$$

From (2.3.1),

$$T^V P \cong P \times \mathfrak{g} \cong \pi^* \mathfrak{g}_P,$$

and from (2.3.4),

$$T^H P \cong \pi^* TM.$$

Therefore, we see that right-invariant horizontal vector fields on P are equivalent to vector fields on M :

$$C^\infty(P, T^H P)^G \cong C^\infty(P, \pi^* TM)^G = \pi^* C^\infty(M, TM);$$

and right-invariant vertical vector fields on P are equivalent to sections of the adjoint bundle \mathfrak{g}_P :

$$C^\infty(P, T^V P)^G \cong C^\infty(P, \pi^* \mathfrak{g}_P)^G = \pi^* C^\infty(M, \mathfrak{g}_P).$$

We also see that right-invariant (i, j) forms on P are equivalent to i -forms on M with values in the j -th exterior power of the dual adjoint bundle:

$$\begin{aligned} \Omega^{i,j}(P)^G &= C^\infty(P, \Lambda^i T^H P^* \otimes \Lambda^j T^V P^*)^G \cong C^\infty(P, \pi^* \Lambda^i TM^* \otimes \pi^* \Lambda^j \mathfrak{g}_P^*)^G \\ &= \pi^* C^\infty(M, \Lambda^i TM^* \otimes \Lambda^j \mathfrak{g}_P^*) = \Omega^i(M; \Lambda^j \mathfrak{g}_P^*). \end{aligned}$$

If $V \in C^\infty(M, TM)$, we will denote the induced right-invariant vector field $\tilde{V} \in C^\infty(P, T^H P)^G$ (and likewise for vertical vector fields). Similarly, for $\psi \in \Omega^i(M; \Lambda^j \mathfrak{g}_P^*)$, we denote the induced form $\tilde{\psi} \in \Omega^{i,j}(P)^G$.

We see that the bi-graded complex $\{\Omega^i(M; \Lambda^j \mathfrak{g}_P^*), d\}$ is isomorphic to $\{\Omega^{i,j}(P)^G, d\}$, but the derivative in the first complex is so far *only* defined by the isomorphism. We now wish to give a more intrinsic description of this differential. Each component of d satisfies the Leibniz rule, so to describe d on $\Omega^i(M; \Lambda^j \mathfrak{g}_P^*)$, it suffices to say how each component acts on $\Omega^i(M)$ and $C^\infty(M; \Lambda^j \mathfrak{g}_P^*)$.

Proposition 3.1.2. *For $d : \Omega^i(M; \mathbb{R}) \rightarrow \Omega^{i+1}(M; \mathbb{R}) \oplus \Omega^i(M; \mathfrak{g}_P^*)$, $d = d^{1,0}$ and is*

the exterior derivative on M , denoted d_M ; i.e. the following diagram is commutative:

$$\begin{array}{ccc} \Omega^i(M) & \xrightarrow{d_M} & \Omega^{i+1}(M) \\ \downarrow \pi^* & & \downarrow \pi^* \oplus 0 \\ \Omega^{i,0}(P)^G & \xrightarrow{d^{1,0}+d^{0,1}} & \Omega^{i+1,0}(P)^G \oplus \Omega^{i,1}(P)^G \end{array}$$

Proof. By naturality of d , we see that

$$d\pi^*\Omega^i(M) = \pi^*d_M\Omega^i(M).$$

This proves commutativity of the diagram. \square

In fact, a converse to the above Proposition also holds.

Lemma 3.1.3. *If $\psi \in \Omega^{i,0}(P)$, and $d^{0,1}\psi = 0$, then $\psi \in \Omega^{i,0}(P)^G$.*

Proof. For $\psi \in \Omega^{i,0}(P)$, the $(0, 1)$ component of d is given by the fiberwise derivative. Let X_1, X_2, \dots, X_i be horizontal right-invariant vector fields on P , and V a vertical vector field on P . Then, the brackets $[V, X_l] = 0$, and

$$d\psi(V, X_1, \dots, X_i) = V\psi(X_1, \dots, X_i).$$

If $d^{0,1}\psi = 0$, then ψ restricted to any fiber $i_x^*\psi \in \Omega^i(P_x)$ is a constant section in $C^\infty(P_x, \pi^*\Lambda^i TM^*)$. Consequently, $\psi \in C^\infty(P, \pi^*\Lambda^i TM^*)^G$. \square

Proposition 3.1.4. $d^{0,1} : \Omega^0(M; \Lambda^j \mathfrak{g}_P^*) \rightarrow \Omega^0(M; \Lambda^{j+1} \mathfrak{g}_P^*)$ is the map induced by the vector bundle homomorphism

$$\Lambda^j \mathfrak{g}_P^* \xrightarrow{d_{\mathfrak{g}}} \Lambda^{j+1} \mathfrak{g}_P^*.$$

In other words, the following diagram is commutative:

$$\begin{array}{ccc} \Omega^0(M, \Lambda^j \mathfrak{g}_P^*) & \xrightarrow{d_{\mathfrak{g}}} & \Omega^0(M, \Lambda^{j+1} \mathfrak{g}_P^*) \\ \downarrow \pi^* & & \downarrow \pi^* \\ \Omega^{0,j}(P)^G & \xrightarrow{d^{0,1}} & \Omega^{0,j+1}(P)^G \end{array}$$

Proof. First, note that the fiberwise inclusion $i_x : P_x \hookrightarrow P$ induces an isomorphism

$$i_{x*} : TP_x \cong (T^V P)_x.$$

Therefore, at any point $p \in P$, we have the canonical isomorphism of complexes

$$(i_x^*)_p : (\Lambda^* T^V P^*)_p \xrightarrow{\cong} (\Lambda^* TP_x^*)_p.$$

For arbitrary $\psi \in \Omega^{0,j}(P)$, $d^{0,1}\psi \in \Omega^{0,j+1}(P)$. This implies

$$(d^{0,1}\psi)_p = i_x^*(d^{0,1}\psi)_p = i_x^*(d\psi)_p = (di_x^*\psi)_p.$$

The middle equality comes from $i_x^*d = i_x^*d^{0,1}$. Therefore, for $\psi = \pi^*\omega \in \Omega^{0,j}(P)^G$, the fiber-wise derivative $(di_x^*)(\pi^*\omega)$ is given by (2.2.4),

$$\begin{array}{ccc} \Lambda^j \mathfrak{g}_{P_x}^* & \xrightarrow{d_{\mathfrak{g}}} & \Lambda^{j+1} \mathfrak{g}_{P_x}^* \\ \downarrow \pi^* & & \downarrow \pi^* \\ \Omega^j(P_x)^G & \xrightarrow{d} & \Omega^{j+1}(P_x)^G \end{array}$$

Therefore, at each point $x \in M$ the section $d^{0,1}\omega$ is given by $d_{\mathfrak{g}}\omega$. \square

Proposition 3.1.5. *The component $d^{1,0} : \Omega^0(M; \Lambda^j \mathfrak{g}_P^*) \rightarrow \Omega^1(M; \Lambda^j \mathfrak{g}_P^*)$ is the co-variant derivative ∇ induced by the connection Θ on the associated bundle $\Lambda^j \mathfrak{g}_P^*$. In other words, the following diagram is commutative:*

$$\begin{array}{ccc} \Omega^0(M; \Lambda^j \mathfrak{g}_P^*) & \xrightarrow{\nabla} & \Omega^1(M; \Lambda^j \mathfrak{g}_P^*) \\ \downarrow \pi^* & & \downarrow \pi^* \\ \Omega^{0,j}(P)^G & \xrightarrow{d^{1,0}} & \Omega^{1,j}(P)^G \end{array}$$

Proof. Let $f \in C^\infty(M; \Lambda^j \mathfrak{g}_P^*)$, and

$$\tilde{f} \in C^\infty(P, \Lambda^j \mathfrak{g}^*)^G \cong \Omega^{0,j}(P)^G$$

be the associated equivariant \mathfrak{g} -valued function. Then,

$$\widetilde{d^{1,0}f} = d^{1,0}\tilde{f}$$

where, in the total space P , $d^{1,0}$ is defined by derivating along horizontal vectors.

Given a horizontal vector field X_M on M , the connection provides a lift to a vector field $\widetilde{X_M}$ on P , and

$$\widetilde{d_{X_M}^{1,0}f} = d_{\widetilde{X_M}}\tilde{f}.$$

As defined in (2.3.5), this is covariant differentiation on the associated bundle induced by the connection Θ . Therefore,

$$\widetilde{\nabla_{X_M}f} = d_{\widetilde{X_M}}\tilde{f} = d^{1,0}\tilde{f}.$$

□

Proposition 3.1.6. $d^{2,-1} : \Omega^0(M; \mathfrak{g}_P^*) \rightarrow \Omega^2(M; \mathbb{R})$ is given by composing the curvature with the natural evaluation; i.e. the following diagram is commutative:

$$\begin{array}{ccccc} \Omega^0(M; \mathfrak{g}_P^*) & \xrightarrow{\Omega} & \Omega^2(M; \mathfrak{g}_P \otimes \mathfrak{g}_P^*) & \longrightarrow & \Omega^2(M; \mathbb{R}) \\ \downarrow \pi^* & & & & \downarrow \pi^* \\ \Omega^{0,1}(P)^G & \xrightarrow{d^{2,-1}} & & \longrightarrow & \Omega^{2,0}(P)^G \end{array}$$

Proof. Recalling (2.3.7), we see that the curvature $\Theta \in \Omega^2(M; \mathfrak{g}_P)$ can be described by taking the Lie bracket of two right-invariant horizontal vector fields and then projecting to the vertical space. If $\widetilde{X_0}, \widetilde{X_1}$ are right-invariant horizontal vector fields

on P , then by (2.3.7)

$$\Omega(\widetilde{X}_0, \widetilde{X}_1) = -\Theta([\widetilde{X}_0, \widetilde{X}_1]) \in C^\infty(P, T^V P)^G.$$

Now, let $\psi \in \Omega^0(M; \mathfrak{g}_P^*)$, and $\widetilde{\psi} = \pi^*\psi \in \Omega^{0,1}(P)^G$. As noted in (3.1.1), $d^{2,-1}$ is determined by

$$d^{2,-1}\widetilde{\psi}(\widetilde{X}_0, \widetilde{X}_1) = -\widetilde{\psi}([\widetilde{X}_0, \widetilde{X}_1]).$$

Because $\widetilde{\psi}$ is zero on horizontal vectors, evaluating it on a vector is the same as first projecting the vector to the vertical tangent space and then evaluating. In other words, for any vector X in TP ,

$$\widetilde{\psi}(X) = \widetilde{\psi}(\Theta(X)).$$

Combining the above equations, we have

$$d^{2,-1}\widetilde{\psi}(\widetilde{X}_0, \widetilde{X}_1) = -\widetilde{\psi}(\Theta([\widetilde{X}_0, \widetilde{X}_1])) = \widetilde{\psi}(\Omega(\widetilde{X}_0, \widetilde{X}_1)) \in \mathbb{R}.$$

Therefore,

$$d^{2,-1}\psi(X_0, X_1) = \psi(\Omega(X_0, X_1))$$

where $\Omega(X_0, X_1) \in \mathfrak{g}_P$ and $\psi \in \mathfrak{g}_P^*$. □

We have now described the $d^{0,1}$, $d^{1,0}$, $d^{2,-1}$ acting on the $\Omega^i(M; \mathbb{R})$ and $\Omega^0(M; \Lambda^j \mathfrak{g}_P^*)$. By using the Leibniz rule, we can extend these descriptions to arbitrary $\Omega^i(M; \Lambda^j \mathfrak{g}_P^*)$. For the following discussion, let $\omega \in \Omega^i(M; \mathbb{R})$ and $\eta \in \Omega^0(M; \Lambda^j \mathfrak{g}_P^*)$. First,

$$\begin{aligned} d^{0,1}(\omega \wedge \eta) &= (d^{0,1}\omega) \wedge \eta + (-1)^i \omega \wedge (d^{0,1}\eta) \\ &= (-1)^i \omega \wedge (d^{0,1}\eta). \end{aligned} \tag{3.1.2}$$

This follows from the Leibniz rule and the fact that $d^{0,1} : \Omega^i(M; \mathbb{R}) \rightarrow \Omega^i(M; \mathfrak{g}_P^*)$ is zero. Therefore, $d^{0,1} : \Omega^i(M; \Lambda^j \mathfrak{g}_P^*) \rightarrow \Omega^i(M; \Lambda^j \mathfrak{g}_P^*)$ is, up to sign, the vector bundle homomorphism induced by $\Lambda^j \mathfrak{g}_P^* \xrightarrow{d_{\mathfrak{g}}} \Lambda^{j+1} \mathfrak{g}_P^*$, and our notation becomes $d^{0,1} =$

$(-1)^i d_{\mathfrak{g}}$;

$$\Omega^i(M; \Lambda^j \mathfrak{g}_P^*) \xrightarrow{(-1)^i d_{\mathfrak{g}}} \Omega^i(M; \Lambda^{j+1} \mathfrak{g}_P^*).$$

Next, we see that

$$\begin{aligned} d^{1,0}(\omega \wedge \eta) &= (d^{1,0}\omega) \wedge \eta + (-1)^i \omega \wedge (d^{1,0}\eta) \\ &= (d_M \omega) \wedge \eta + (-1)^i \omega \wedge (\nabla \eta). \end{aligned}$$

Here, we use the general symbol ∇ for the connection on each bundle $\Lambda^j \mathfrak{g}_P^*$, as the bundles and connections are all constructed from (P, Θ) . The $d^{0,1}$ satisfies

- $d^{1,0}\eta = \nabla \eta$ for $\eta \in \Omega^0(M; \Lambda^j \mathfrak{g}_P^*)$,
- $d^{1,0}(\omega \wedge \eta) = (d_M \omega) \wedge \eta + (-1)^i \omega \wedge (\nabla \eta)$.

Therefore (2.3.6), $d^{1,0}$ is the standard extension of $\nabla : \Omega^0(M; \Lambda^j \mathfrak{g}_P^*) \rightarrow \Omega^1(M; \Lambda^j \mathfrak{g}_P^*)$ to the complex $\{\Omega^i(M; \Lambda^j \mathfrak{g}_P^*)\}$. We therefore use the notation $d^{1,0} = d_{\nabla}$, and

$$\Omega^0(M; \Lambda^j \mathfrak{g}_P^*) \xrightarrow{\nabla} \Omega^1(M; \Lambda^j \mathfrak{g}_P^*) \xrightarrow{d_{\nabla}} \Omega^2(M; \Lambda^j \mathfrak{g}_P^*) \xrightarrow{d_{\nabla}} \dots$$

To write $d^{2,-1}$ on arbitrary sections in $\Omega^0(M; \Lambda^j \mathfrak{g}_P^*)$, we observe that $d^{2,-1}$ satisfies the properties

- $d^{2,-1}\eta = \eta(\Omega)$ for $\eta \in \Omega^{0,1}$,
- $d^{2,-1}(\eta_1 \wedge \eta_2) = (d^{2,-1}\eta_1) \wedge \eta_2 + (-1)^{\deg \eta_1} \eta_1 \wedge (d^{2,-1}\eta_2)$.

Therefore, $d^{2,-1} : \Omega^0(M; \Lambda^j \mathfrak{g}_P^*) \rightarrow \Omega^2(M; \Lambda^{j-1} \mathfrak{g}_P^*)$ is simply the contraction along the (vertical-vector)-valued 2-form Ω . We denote this by ι_{Ω} . Evaluating on elements of $\Omega^i(M; \Lambda^j \mathfrak{g}_P^*)$,

$$\begin{aligned} d^{2,-1}(\omega \wedge \eta) &= (-1)^i \omega \wedge (d^{2,-1}\eta) \\ &= (-1)^i \omega \wedge (\iota_{\Omega} \eta). \end{aligned}$$

We use the notation $d^{2,-1} = (-1)^i \iota_\Omega$, and

$$\Omega^i(M; \Lambda^j \mathfrak{g}_P^*) \xrightarrow{(-1)^i \iota_\Omega} \Omega^{i+2}(M; \Lambda^{j-1} \mathfrak{g}_P^*).$$

Combining the above descriptions of the components of d gives the following proposition.

Proposition 3.1.7. *The following bi-graded cochain complex is isomorphic to $\{\Omega^{i,j}(P)^G, d\}$, the cochain complex of right-invariant forms on P .*

$$\begin{array}{ccccccccccc}
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
& \uparrow d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} & & \uparrow d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} & \\
0 & \longrightarrow & \Omega^0(M; \Lambda^2 \mathfrak{g}_P^*) & \xrightarrow{\nabla} & \Omega^1(M; \Lambda^2 \mathfrak{g}_P^*) & \xrightarrow{d_{\nabla}} & \Omega^2(M; \Lambda^2 \mathfrak{g}_P^*) & \xrightarrow{d_{\nabla}} & \Omega^3(M; \Lambda^2 \mathfrak{g}_P^*) & \xrightarrow{d_{\nabla}} & \cdots \\
& \uparrow d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} & & \uparrow d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} & \\
0 & \longrightarrow & \Omega^0(M; \mathfrak{g}_P^*) & \xrightarrow{\nabla} & \Omega^1(M; \mathfrak{g}_P^*) & \xrightarrow{d_{\nabla}} & \Omega^2(M; \mathfrak{g}_P^*) & \xrightarrow{d_{\nabla}} & \Omega^3(M; \mathfrak{g}_P^*) & \xrightarrow{d_{\nabla}} & \cdots \\
& \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & \\
0 & \longrightarrow & \Omega^0(M; \mathbb{R}) & \xrightarrow{d_M} & \Omega^1(M; \mathbb{R}) & \xrightarrow{d_M} & \Omega^2(M; \mathbb{R}) & \xrightarrow{d_M} & \Omega^3(M; \mathbb{R}) & \xrightarrow{d_M} & \cdots \\
& \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & \\
& & & & & & & & & &
\end{array}$$

Remark 3.1.8. *The above complex is not a commutative diagram, and $d_{\nabla}^2 \neq 0$.*

Instead, we have that

$$0 = d^2 = (d^{0,1} + d^{1,0} + d^{2,-1})^2,$$

and therefore we see the equalities

$$\begin{aligned}
0 &= (d^{0,1})^2 = d_{\mathfrak{g}}^2 \\
0 &= d^{0,1}d^{1,0} + d^{1,0}d^{0,1} = \pm(d_{\mathfrak{g}}d_{\nabla} - d_{\nabla}d_{\mathfrak{g}}) \\
0 &= d^{0,1}d^{2,-1} + d^{1,0}d^{1,0} + d^{2,-1}d^{0,1} = d_{\mathfrak{g}}\iota_{\Omega} + d_{\nabla}^2 + \iota_{\Omega}d_{\mathfrak{g}} \\
0 &= d^{1,0}d^{2,-1} + d^{2,-1}d^{1,0} = \pm(d_{\nabla}\iota_{\Omega} - \iota_{\Omega}d_{\nabla}) \\
0 &= (d^{2,-1})^2 = \iota_{\Omega}^2
\end{aligned}$$

In other words, if you start at a point in the diagram and look at another point “two arrows away,” summing over all the possible paths between the two points gives zero. From this point of view, the bi-graded complex resembles a spectral sequence.

3.2 The adjoint d^* in the bi-graded complex

For Chapter 5 of this thesis, we wish to think of P as a Riemannian manifold and study the low-dimensional harmonic forms. Given a metric g on M and a connection Θ on P , along with a bi-invariant metric g_G on G , we define a canonical right-invariant metric g_P on P . The metric g_P is defined by g on the horizontal tangent bundle $T^H P$, and g_G on the vertical tangent bundle $T^V P$. However, we are really interested in the 1-parameter family of metrics g_{δ} on P given by g on $T^H P$ and $\delta^2 g_G$ on $T^V P$. The results in Chapter 5, where we look at the limit of g_{δ} -harmonic forms as $\delta \rightarrow 0$, will be independent of the choice of metric g_G on the fibers G . Since each metric g_{δ} is the pullback of metrics on TM and \mathfrak{g}_P , the inner product space of G -invariant forms on P is naturally isomorphic to the inner product space $\{\Omega^i(M; \Lambda^j \mathfrak{g}_P^*), \Lambda^i g^* \otimes \Lambda^j g_G^*\}$. This implies that when restricted to right-invariant forms, the adjoint $d^{*g_{\delta}}$ is given by the adjoints to $d_{\mathfrak{g}}$, d_M , d_{∇} , and ι_{Ω} .

We now proceed in more detail. Start with the data in the previous sections:

(M, P, Θ) where $P \xrightarrow{\pi} M$ is a principal G -bundle with connection Θ . In fact,

$$TP = T^H P \oplus T^V P = \pi^*(TM \oplus \mathfrak{g}_P),$$

where $\mathfrak{g}_P \rightarrow M$ is the adjoint bundle $P \times_{Ad} \mathfrak{g}$. We introduce the choice of a Riemannian metric g on M (which is a metric on the bundle TM). This is denoted (M, P, g, Θ) . Next, we must choose an Ad-invariant metric g_G on \mathfrak{g} . Since G is compact, such metrics always exist. In fact, there is a 1-1 correspondence

$$\{\text{Bi-invariant metrics on } G\} \longleftrightarrow \{\text{Ad-invariant metrics on } \mathfrak{g}\},$$

and we will sometimes blur the distinction between the two. Since g_G is Ad-invariant, this induces a metric on the adjoint bundle $\mathfrak{g}_P \rightarrow M$, which we also denote g_G .

We now have a natural right-invariant metric g_P on P given by

$$g_P \stackrel{\text{def}}{=} \pi^*(g \oplus g_G).$$

In other words, the metric on the horizontal spaces is defined by lifting the metric from the base to a metric on $T^H P$, the metric on the fibers is given by the metric on G , and the horizontal and vertical subspaces are defined to be perpendicular. Calculating at a point, this looks like

$$\begin{aligned} (TP, g_P) &\stackrel{\text{def}}{=} (T^H P, g) \oplus (T^V P, g_G) \\ g_P(v_1, v_2) &= g(\pi_* v_1, \pi_* v_2) + g_G(\Theta(v_1), \Theta(v_2)). \end{aligned}$$

The metric on g_P induces metrics on all the spaces of forms

$$(\Omega^{i,j}(P), \pi^*(\Lambda^i g^* \otimes \Lambda^j g_G^*)).$$

Restricting to the subcomplex of right-invariant forms, we have the following iso-

morphisms of cochain complexes with metric:

$$\begin{aligned}
(\Omega^{i,j}(P), \pi^*(\Lambda^i g^* \otimes \Lambda^j g_G^*))^G &\cong (C^\infty(P, \pi^*(\Lambda^i TM^* \otimes \Lambda^j \mathfrak{g}_P^*)), \pi^*(\Lambda^i g^* \otimes \Lambda^j g_G^*))^G \\
&= \pi^*(C^\infty(M, \Lambda^i TM^* \otimes \Lambda^j \mathfrak{g}_P^*), \Lambda^i g^* \otimes \Lambda^j g_G^*) \\
&= \pi^*(\Omega^i(M; \Lambda^j \mathfrak{g}_P^*), \Lambda^i g^* \otimes \Lambda^j g_G^*).
\end{aligned}$$

All of the metrics are induced from the metrics g and g_G , and we will always think of g_G as fixed, while g can be any element in $Met(M)$.

As defined in (2.4.1), let

$$d^{*g_P} : \Omega^k(P) \rightarrow \Omega^{k-1}(P)$$

be the adjoint to d on $\Omega^*(P)$, and let d^* denote the adjoint to the differential d on $\Omega^i(M; \Lambda^j \mathfrak{g}_P^*)$. Then, using the inclusion of cochain complexes with metric

$$\{\Omega^{i,j}(P)^G, d\} \hookrightarrow \{\Omega^{i,j}(P), d\},$$

we have the induced inclusion on the dual complexes

$$(\Omega^i(M; \Lambda^j \mathfrak{g}_P^*), d^*) \cong (\Omega^{i,j}(P)^G, d^{*g_P}) \hookrightarrow (\Omega^{i,j}(P), d^{*g_P}). \quad (3.2.1)$$

Under the bi-grading,

$$d^* = d^{0,1*} + d^{1,0*} + d^{2,-1*},$$

where

$$d^{a,b*} : \Omega^i(M; \Lambda^j \mathfrak{g}_P^*) \rightarrow \Omega^{i-a}(M; \Lambda^{j-b} \mathfrak{g}_P^*).$$

Using our description of the differentials $d^{a,b}$ in Proposition 3.1.7, we write this as

$$d^{0,1*} = (-1)^i (d_{\mathfrak{g}})^*, \quad d^{1,0*} = (d_{\nabla})^*, \quad d^{2,-1*} = (-1)^i (\iota_{\Omega})^*.$$

where the adjoints are induced by the metrics g and g_G on TM and \mathfrak{g}_P , respectively.

We therefore have the following complex on right-invariant forms determined by the coderivative associated to the right-invariant metric g_P .

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & & \vdots \\
& \downarrow \iota_{\Omega}^* & & \downarrow -\iota_{\Omega}^* & & \downarrow d_{\mathfrak{g}}^* & & \downarrow -d_{\mathfrak{g}}^* \\
0 \longleftarrow & \Omega^0(M; \Lambda^2 \mathfrak{g}_P^*) & \xleftarrow{d_{\nabla}^*} & \Omega^1(M; \Lambda^2 \mathfrak{g}_P^*) & \xleftarrow{d_{\nabla}^*} & \Omega^2(M; \Lambda^2 \mathfrak{g}_P^*) & \xleftarrow{d_{\nabla}^*} & \Omega^3(M; \Lambda^2 \mathfrak{g}_P^*) & \xleftarrow{d_{\nabla}^*} \cdots \\
& \downarrow d_{\mathfrak{g}}^* & & \downarrow -d_{\mathfrak{g}}^* & & \downarrow d_{\mathfrak{g}}^* & & \downarrow -d_{\mathfrak{g}}^* \\
0 \longleftarrow & \Omega^0(M; \mathfrak{g}_P^*) & \xleftarrow{d_{\nabla}^*} & \Omega^1(M; \mathfrak{g}_P^*) & \xleftarrow{d_{\nabla}^*} & \Omega^2(M; \mathfrak{g}_P^*) & \xleftarrow{d_{\nabla}^*} & \Omega^3(M; \mathfrak{g}_P^*) & \xleftarrow{d_{\nabla}^*} \cdots \\
& \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 \\
0 \longleftarrow & \Omega^0(M; \mathbb{R}) & \xleftarrow{d_M^*} & \Omega^1(M; \mathbb{R}) & \xleftarrow{d_M^*} & \Omega^2(M; \mathbb{R}) & \xleftarrow{d_M^*} & \Omega^3(M; \mathbb{R}) & \xleftarrow{d_M^*} \cdots \\
& \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0
\end{array}$$

In particular, the adjoint $d_M^* : \Omega^i(M; \mathbb{R}) \rightarrow \Omega^{i-1}(M; \mathbb{R})$ is the usual adjoint with respect to the metric g , and the adjoint $d_{\mathfrak{g}}^*$ is induced from the usual adjoint to the Lie algebra derivative with respect to the metric g_G .

3.3 Rescaled derivatives d_{δ} and d_{δ}^*

In general, calculating the harmonic forms on P is still very difficult. It turns out that if we shrink the metric on the fiber while keeping the metric on the base the same, the harmonic forms become easier to compute and have an interesting description. This limiting process is referred to as the adiabatic limit. We now define the 1-parameter family of metrics g_{δ} , the rescaled derivative d_{δ} , the rescaled coderivative d_{δ}^* , and the rescaled Laplacian $L_{g_{\delta}}$ which is an operator on the fixed inner product space

$$L_{g_{\delta}}^p : (\Omega^p(P), g_P) \rightarrow (\Omega^p(P), g_P).$$

Remark 3.3.1. *The standard notation in the literature is $g_{\delta} = \delta^{-2}g \oplus g_G$ with $\delta \rightarrow 0$, or expanding the base relative to the fibers. Since harmonic forms remain*

invariant under a global rescaling, this is equivalent to shrinking the fiber relative to the base. The notation used from this point on will coincide with the literature, but the author finds it conceptually more convenient to think of this as shrinking the fiber.

As before, we choose an Ad -invariant metric g_G on \mathfrak{g} (or a bi-invariant metric on the fiber G). Just as we defined a metric $g_P = \pi^*(g \oplus g_G)$, define a 1-parameter family of metrics on P by

$$g_\delta \stackrel{\text{def}}{=} \pi^*(\delta^{-2}g \oplus g_G). \quad (3.3.1)$$

For any $\delta > 0$, g_δ is a right-invariant Riemannian metric on P . Because the inner product space $\Omega^*(P, g_\delta)$ is changing for each δ , we have a 1-parameter family of inner product spaces and operators. It is therefore convenient to fix the inner product by introducing the isometry

$$\begin{aligned} \rho_\delta: (\Omega^{i,j}(P), g_\delta) &\rightarrow (\Omega^{i,j}(P), g_P) \\ \phi &\mapsto \delta^i \phi \end{aligned}$$

Conjugating by this isometry gives a 1-parameter family of operators on the fixed inner product space $(\Omega^*(P), g_P)$. Let Δ_{g_δ} be the usual Hodge Laplacian

$$\Delta_{g_\delta}^p \stackrel{\text{def}}{=} dd^*_{g_\delta} + d^*_{g_\delta}d : (\Omega^p(P), g_\delta) \rightarrow (\Omega^p(P), g_\delta),$$

and define the operators d_δ, d_δ^* , and L_δ on the space $(\Omega^*(P), g_P)$ by

$$\begin{aligned} d_\delta &\stackrel{\text{def}}{=} \rho_\delta d \rho_\delta^{-1} \\ d_\delta^* &\stackrel{\text{def}}{=} \rho_\delta d^*_{g_\delta} \rho_\delta^{-1} \\ L_{g_\delta} &\stackrel{\text{def}}{=} \rho_\delta \Delta_{g_\delta} \rho_\delta^{-1} = d_\delta d_\delta^* + d_\delta^* d_\delta \end{aligned}$$

In general, rescaling a metric by a global factor δ^2 introduces powers of δ on the $*$ operator (see Lemma 2.4.5). Therefore, while d remains invariant, d^* picks up powers of δ . In our situation, combined with the bi-grading, this looks like

$$\begin{aligned} d &= d^{0,1} + d^{1,0} + d^{2,-1} \\ d^{*g_\delta} &= d^{0,1*} + \delta^2 d^{1,0*} + \delta^4 d^{2,-1*} \end{aligned}$$

where $*$ denotes the adjoint with respect to the original metric. Since the conjugation action also introduces powers of δ , then the components of both d_δ and d_δ^* pick up powers of δ according to the bi-grading:

$$\begin{aligned} d_\delta &= d^{0,1} + \delta d^{1,0} + \delta^2 d^{2,-1} \\ d_\delta^* &= d^{0,1} + \delta d^{1,0*} + \delta^2 d^{2,-1*} \end{aligned}$$

When applied to the complex of right-invariant forms, which is isomorphic to $\{\Omega^i(M; \Lambda^j \mathfrak{g}_P^*)\}$ (as described in Proposition 3.1.7 and (3.3.3))

$$\begin{aligned} d_\delta &= \pm d_{\mathfrak{g}} + \delta d_{\nabla} \pm \delta^2 \iota_\Omega \\ d_\delta^* &= \pm d_{\mathfrak{g}}^* + \delta d_{\nabla}^* \pm \delta^2 \iota_\Omega^* \end{aligned}$$

Therefore, $\{\Omega^{i,j}(P)^G, d_\delta\}$ is isomorphic to the cochain complex

$$\begin{array}{ccccccccc}
& \vdots & & \vdots & & \vdots & & \vdots & \\
& \uparrow d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} & & \uparrow d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} & \\
0 & \longrightarrow & \Omega^0(M; \Lambda^2 \mathfrak{g}_P^*) & \xrightarrow{\delta^\nabla} & \Omega^1(M; \Lambda^2 \mathfrak{g}_P^*) & \xrightarrow{\delta d^\nabla} & \Omega^2(M; \Lambda^2 \mathfrak{g}_P^*) & \xrightarrow{\delta d^\nabla} & \Omega^3(M; \Lambda^2 \mathfrak{g}_P^*) \xrightarrow{\delta d^\nabla} \dots \\
& \uparrow d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} & & \uparrow d_{\mathfrak{g}} & & \uparrow -d_{\mathfrak{g}} & \\
0 & \longrightarrow & \Omega^0(M; \mathfrak{g}_P^*) & \xrightarrow{\delta^\nabla} & \Omega^1(M; \mathfrak{g}_P^*) & \xrightarrow{\delta d^\nabla} & \Omega^2(M; \mathfrak{g}_P^*) & \xrightarrow{\delta d^\nabla} & \Omega^3(M; \mathfrak{g}_P^*) \xrightarrow{\delta d^\nabla} \dots \\
& \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & \\
0 & \longrightarrow & \Omega^0(M; \mathbb{R}) & \xrightarrow{\delta d_M} & \Omega^1(M; \mathbb{R}) & \xrightarrow{\delta d_M} & \Omega^2(M; \mathbb{R}) & \xrightarrow{\delta d_M} & \Omega^3(M; \mathbb{R}) \xrightarrow{\delta d_M} \dots \\
& \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & \\
& & & & & & & &
\end{array} \tag{3.3.2}$$

Likewise, the dual complex $\{\Omega^{i,j}(P)^G, d_\delta^*\}$ is isomorphic to

$$\begin{array}{ccccccccc}
& \vdots & & \vdots & & \vdots & & \vdots & \\
& \downarrow \delta^2 \iota_\Omega^* & & \downarrow -\delta^2 \iota_\Omega^* & & \downarrow d_{\mathfrak{g}}^* & & \downarrow -d_{\mathfrak{g}}^* & \\
0 & \longleftarrow & \Omega^0(M; \Lambda^2 \mathfrak{g}_P^*) & \xleftarrow{\delta d_\nabla^*} & \Omega^1(M; \Lambda^2 \mathfrak{g}_P^*) & \xleftarrow{\delta d_\nabla^*} & \Omega^2(M; \Lambda^2 \mathfrak{g}_P^*) & \xleftarrow{\delta d_\nabla^*} & \Omega^3(M; \Lambda^2 \mathfrak{g}_P^*) \xleftarrow{\delta d_\nabla^*} \dots \\
& \downarrow d_{\mathfrak{g}}^* & & \downarrow -d_{\mathfrak{g}}^* & & \downarrow d_{\mathfrak{g}}^* & & \downarrow -d_{\mathfrak{g}}^* & \\
0 & \longleftarrow & \Omega^0(M; \mathfrak{g}_P^*) & \xleftarrow{\delta d_\nabla^*} & \Omega^1(M; \mathfrak{g}_P^*) & \xleftarrow{\delta d_\nabla^*} & \Omega^2(M; \mathfrak{g}_P^*) & \xleftarrow{\delta d_\nabla^*} & \Omega^3(M; \mathfrak{g}_P^*) \xleftarrow{\delta d_\nabla^*} \dots \\
& \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 & \\
0 & \longleftarrow & \Omega^0(M; \mathbb{R}) & \xleftarrow{\delta d_M^*} & \Omega^1(M; \mathbb{R}) & \xleftarrow{\delta d_M^*} & \Omega^2(M; \mathbb{R}) & \xleftarrow{\delta d_M^*} & \Omega^3(M; \mathbb{R}) \xleftarrow{\delta d_M^*} \dots \\
& \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 & \\
& & & & & & & &
\end{array} \tag{3.3.3}$$

These two complexes will be very important in the calculations of Chapter 5. For the reader's convenience, these two pictures are reproduced on an extra page at the end of the thesis.

CHAPTER 4

ADIABATIC SPECTRAL SEQUENCE

Our goal in the next chapter is to calculate, for $k = 1, 2, 3$, the harmonic k -forms on P in the adiabatic limit. In other words, if $\Delta_{g_\delta}^k : (\Omega^k(P), g_\delta) \rightarrow (\Omega^k(P), g_\delta)$ is the Hodge Laplacian, then what does

$$\lim_{\delta \rightarrow 0} \text{Ker } \Delta_{g_\delta}^k$$

look like? The most obvious question is whether this even makes sense. In fact, there is a smooth extension of $\text{Ker } \Delta_{g_\delta}$ to $\delta = 0$ ([MM]). The more important question for us is how to calculate this limit. This comes from interpreting $\lim_{\delta \rightarrow 0} \text{Ker } \Delta_{g_\delta}$ in terms of a spectral sequence. Roughly, given a form $\omega \in \Omega^p(P)$, we say that $\omega \in E_K^p$ if there is a 1-parameter family of forms $\omega(\delta)$ on the closed interval $[0, 1]$ such that $\omega(0) = \omega$, and both $d_\delta \omega(\delta)$ and $d_\delta^* \omega(\delta)$ vanish to order K as δ goes to 0. In other words, if ω can be extended to a family of forms that is d_δ -closed and d_δ^* -coclosed up to order K at $\delta = 0$, then $\omega \in E_K^p$. This filtration on the space of forms is then isomorphic to the Leray–Serre spectral sequence.

This idea was developed by Mazzeo and Melrose in [MM] and expanded in [Dai] and [For]. In particular, Forman made explicit the spectral sequence structure, including the differentials. The following is a summary of his treatment. The spectral sequence structure holds for more general situations than ours, so we start with a more general discussion, and add in extra hypotheses as needed. In particular,

all statements will hold for our situation of a principal bundle with metric given by (M, P, g, Θ) .

Remark 4.0.2. *Warning! In Forman’s paper, his bi-grading notation is opposite the notation here. Whereas our notation is always (horizontal, vertical), Forman’s notation is (vertical, horizontal). This also means that the distributions A and B are reversed. In the specific example we look at, the bi-grading naturally shows up in the notation $\Omega^i(M; \Lambda^j \mathfrak{g}_P^*)$, and it seems more natural to call this space $\Omega^{i,j}(P)$ than the opposite.*

4.1 Leray–Serre spectral sequence

In general, a cohomology spectral sequence is comprised of

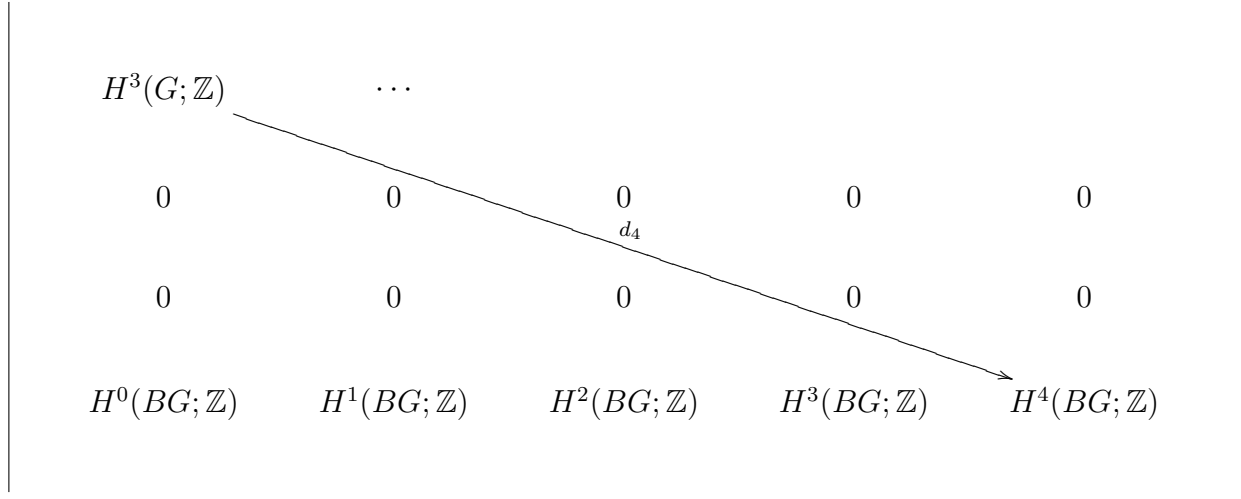
- a sequence of “pages” or cochain complexes $E_0^p, E_1^p, E_2^p, \dots, E_\infty^p$. Frequently, these are equipped with a bi-grading $E_K^p = \bigoplus_{i+j=p} E_K^{i,j}$.
- differentials $d_K : E_K^p \rightarrow E_{K+1}^{p+1}$ such that $H^p(E_K^*, d_K) = E_{K+1}^p$. If there is a bi-grading present, then $d_K : E_K^{i,j} \rightarrow E_{K+1}^{i+K, j-K+1}$.

Normally, there are isomorphisms relating the E_2 and E_∞ pages to something with a more intrinsic description. The example we deal with is the Leray–Serre spectral sequence associated to a fibration of compact manifolds $F \hookrightarrow P \xrightarrow{\pi} M$. Then,

- $E_2^{i,j} \cong H^i(M; H^j(F; H))$ (if $\pi_1(M)$ acts trivially on P),
- E_∞^p gives $H^p(M; H)$ in the sense that $\{E_\infty^{n,p-n}\}_n$ is isomorphic to the quotients F_n^p / F_{n+1}^p of a filtration $0 \subset F_p^p \subset \dots \subset F_0^p = H^p(P; H)$.

In other words, we have a method of relating the cohomology of the total space of a fibration with the cohomology of its base and fiber. This allows for powerful calculations using very formal methods. For example, ...

Example 4.1.1. *Let's apply the Leray–Serre spectral sequence for integral cohomology to the universal principal G -bundle $EG \rightarrow BG$, where G is compact, simple, and simply-connected. Since $H^1(G; \mathbb{Z}) = H^2(G; \mathbb{Z}) = 0$, and $H^3(G; \mathbb{Z}) \cong \mathbb{Z}$, then, the E_2 through E_4 pages look like the following, with the only non-trivial differential being d_4 on the E_4 page.*



EG is a contractible space, and therefore the E_∞ page must be zero (except for $E_\infty^{0,0} \cong H^0(EG; \mathbb{Z}) \cong \mathbb{Z}$). Consequently, $H^i(BG; \mathbb{Z}) = 0$ for $1 \leq i \leq 3$, and

$$d_4 : H^3(G; \mathbb{Z}) \xrightarrow{\cong} H^4(BG; \mathbb{Z}).$$

The generator of $H^3(G; \mathbb{Z})$ gets sent to a universal 4-dimensional characteristic class. For $G = SU(n)$ ($n \geq 2$) this is the c_2 class, and for $G = Spin(n)$ ($n \geq 5$) this is the $\frac{p_1}{2}$ class.

4.2 Adiabatic spectral sequence

The following spectral sequence is defined in a very general situation, and the treatment is taken almost directly from Section 0 of [For]. Once we impose certain geometric conditions, this gives analytic information. Suppose we have a compact

Riemannian manifold (P, g_P) (not necessarily a principal bundle) with two orthogonal distributions A and B in TP such that

$$A \oplus B = TP$$

and $g_P = g_A \oplus g_B$. Neither A nor B need to be integrable. This induces a bi-grading on the space of differential forms

$$\Omega^p(P) = \bigoplus_{i+j=p} \Omega^{i,j}(P), \quad \Omega^{i,j}(P) = C^\infty(P, \Lambda^i A^* \oplus \Lambda^j B^*);$$

the exterior derivative and its dual decompose as

$$d = d^{-1,2} + d^{0,1} + d^{1,0} + d^{2,-1}, \quad d^{a,b} : \Omega^{i,j}(P) \rightarrow \Omega^{i+a,j+b}(P),$$

$$d^{*g} = (d^{-1,2})^* + (d^{0,1})^* + (d^{1,0})^* + (d^{2,-1})^*, \quad d^{a,b*} : \Omega^{i,j}(P) \rightarrow \Omega^{i-a,j-b}(P).$$

Definition 4.2.1. *Define the 1-parameter family of metrics g_δ by*

$$g_\delta \stackrel{\text{def}}{=} \delta^{-2} g_A \oplus g_B.$$

Introducing the isometry (for any $\delta > 0$)

$$\rho_\delta : (\Omega^{i,j}, g_\delta) \rightarrow (\Omega^{i,j}, g_P)$$

$$\phi \mapsto \delta^i \phi$$

gives rise to operators d_δ, d_δ^* , and L_{g_δ} on the space $(\Omega^*(P), g_P)$ defined by

$$d_\delta \stackrel{\text{def}}{=} \rho_\delta d \rho_\delta^{-1}, \quad d_\delta = \delta^{-1} d^{-1,2} + d^{0,1} + \delta d^{1,0} + \delta^2 d^{2,-1}, \quad (4.2.1)$$

$$d_\delta^* \stackrel{\text{def}}{=} \rho_\delta d^{*g_\delta} \rho_\delta^{-1}, \quad d_\delta^* = \delta^{-1} d^{-1,2*} + d^{0,1*} + \delta d^{1,0*} + \delta^2 d^{2,-1*}, \quad (4.2.2)$$

$$L_{g_\delta} \stackrel{\text{def}}{=} \rho_\delta \Delta_{g_\delta} \rho_\delta^{-1} = d_\delta d_\delta^* + d_\delta^* d_\delta. \quad (4.2.3)$$

In addition to fixing the Hilbert space, the isometry ρ_δ produces the above factors of δ , which naturally give the spectral sequence structure. First, we define the terms $E_K^{i,j}$ and then give the differentials and show why this is a spectral sequence. This is the opposite of what normally occurs, since a spectral sequence is usually defined inductively. In dealing with $\{E_K^{i,j}\}$ as a spectral sequence, it is useful to treat δ as a formal variable. However, from a geometric point of view, δ is the value of a parameter. Note that these two points of view agree on polynomials in $\Omega^*(P)[\delta]$. In other words, d_δ and d_δ^* act formally on $\Omega^*(P)[\delta]$, but if we evaluate at some $\delta = 0$, then this agrees with the actual d_δ and d_δ^* acting on $\Omega^*(P)$. This is most conveniently expressed in the commutative diagram, where ev_δ denotes the evaluation map at some $\delta > 0$.

$$\begin{array}{ccc} \Omega^*(P)[\delta] & \xrightarrow{d_\delta} & \Omega^*(P)[\delta] \\ ev_\delta \downarrow & & ev_\delta \downarrow \\ \Omega^*(P) & \xrightarrow{d_\delta} & \Omega^*(P) \end{array}$$

We will also deal with infinite formal powers series, and we will then be careful to make the distinction between L_{g_δ} defined in (4.2.3) and the formal

$$L_\delta : \Omega^*(P)[[\delta]] \rightarrow \Omega^*(P)[[\delta]] \tag{4.2.4}$$

given by formally applying $d_\delta^*d_\delta + d_\delta d_\delta^*$.

Definition 4.2.2.

$$\begin{aligned} E_K^{i,j} &\stackrel{def}{=} \{\omega \in \Omega^{i,j}(P) \mid \exists \omega_1, \dots, \omega_l \in \Omega^{i+j}(P) \text{ with} \\ &\quad d_\delta(\omega + \delta\omega_1 + \dots + \delta^l\omega_l) \in \delta^K \Omega^*(P)[\delta] \\ &\quad d_\delta^*(\omega + \delta\omega_1 + \dots + \delta^l\omega_l) \in \delta^K \Omega^*(P)[\delta]\} \end{aligned}$$

The above definition is very formal, and hence useful for calculations. However,

we could think of this in a more geometric way. Let $\Omega^{i+j}(P) \rightarrow [0, 1]$ be the trivial (infinite-dimensional) vector bundle over the interval $[0, 1]$. The polynomial $\omega + \delta\omega_1 + \cdots + \delta^l\omega_l$ can be thought of as the Taylor approximation at $\delta = 0$ of a section

$$\tilde{\omega} \in C^\infty([0, 1], \Omega^{i+j}(P)).$$

Since for any $\delta > 0$ we have the operators d_δ and d_δ^* , we can give an equivalent definition

$$E_K^{i,j} = \{\omega \in \Omega^{i,j}(P) \mid \exists \tilde{\omega} \in C^\infty([0, 1], \Omega^{i+j}(P)) \text{ such that } \tilde{\omega}(0) = \omega$$

$$\text{and as } \delta \rightarrow 0, d_\delta \tilde{\omega} = 0 + O(\delta^K); d_\delta^* \tilde{\omega} = 0 + O(\delta^K)\}$$

By definition, $\Omega^{i,j}(P) = E_{-1}^{i,j} \supseteq E_0^{i,j} \cdots \supseteq E_K^{i,j} \supseteq E_{K+1}^{i,j} \supseteq \cdots \supseteq E_\infty^{i,j}$. Thus, we have a filtration on $\Omega^{i,j}(P)$. (The E_{-1} term is included, since d_δ and d_δ^* have a negative power of δ if the distribution B is not integrable.) The space $E_\infty^{i,j}$ is defined by

$$E_\infty^{i,j} \stackrel{\text{def}}{=} \bigcap_K E_K^{i,j},$$

and can be described by forms $\omega \in \Omega^{i,j}(P)$ that, when viewed as a boundary value at $\delta = 0$ in the space of sections $C^\infty([0, 1], \Omega^{i+j}(P))$, can be extended so that d_δ and d_δ^* of the extension vanish up to any order of δ .

We now focus on the spectral sequence structure, and consequently we deal with δ formally. To define the differentials which make $\{E_K^{i,j}\}$ into a spectral sequence, first let

$$\pi_K : \Omega^*(P) \rightarrow E_K$$

denote the orthogonal projection using the metric g . Suppose $\omega \in E_K$. Then, we know there exists a (non-unique) extension $\omega_\delta = \omega + \delta\omega_1 + \cdots$ such that $d_\delta \omega_\delta, d_\delta^* \omega_\delta \in$

$\delta^K \Omega^*(P)[\delta]$. To any extension ω_δ , define

$$d_K \omega_\delta \stackrel{\text{def}}{=} \lim_{\delta \rightarrow 0} \delta^{-K} d_\delta \omega_\delta = \lim_{\delta \rightarrow 0} \delta^{-K} d_\delta (\omega + \delta \omega_1 + \delta^2 \omega_2 + \dots).$$

Though this depends on the choice of extension,

$$\pi_K d_K \pi_K : E_K \rightarrow E_K$$

is well-defined. The same thing can also be done to define

$$\pi_K d_K^* \pi_K : E_K \rightarrow E_K.$$

We then form the second order operator

$$\Delta_K \stackrel{\text{def}}{=} (\pi_K d_K \pi_K) (\pi_K d_K^* \pi_K) + (\pi_K d_K^* \pi_K) (\pi_K d_K \pi_K).$$

Theorem 4.2.3. (*Theorem 2.5 [For]*)

1. $(\pi_K d_K \pi_K)^2 = (\pi_K d_K^* \pi_K)^2 = 0.$
2. $\text{Ker } \Delta_K = \text{Ker}(\pi_K d_K \pi_K) \cap \text{Ker}(\pi_K d_K^* \pi_K) = E_{K+1}.$
3. $\pi_K d_K \pi_K : E_K^{i,j} \rightarrow E_K^{i+K, j-K+1}.$

The maps $\pi_K d_K \pi_K$ form differentials in the complex E_K^\bullet . However, instead of taking the ordinary cohomology and dealing with equivalence classes of cocycles, we take the Hodge cohomology of the complex E_K^\bullet . Therefore, $E_{K+1}^\bullet \subset \Omega^*(P)$; the cochains of the new complex are still represented by differential forms, and the sequence of complexes $\{E_K^{i,j}, \pi_K d_K \pi_K\}$ form a spectral sequence.

Again, this spectral sequence is defined for arbitrary orthogonal distributions A and B such that $A \oplus B = TP$. In general, one can describe the convergence of this sequence using the language of Laurent cohomology. Since this is not important for

our later discussion, we don't go into details. Roughly speaking, though, the E_∞^p term gives a basis for $\text{Ker } L_\delta$ acting formally on Laurent series with differential forms as coefficients. This is then isomorphic to formal Laurent series with coefficients in $H^p(P)$. In particular, $\dim E_\infty^p = \dim H^p(P)$, where E_∞^p is a finite-dimensional subspace of differential forms. However, when B is integrable, we have the following isomorphism.

Proposition 4.2.4. *(Corollary 4.4 [For]) If the distribution B is integrable, then the spectral sequence $\{E_K^{i,j}, \pi_K d_K \pi_K\}$ is isomorphic to the standard Leray–Serre spectral sequence for the foliation associated to B .*

Before discussing the relationship with harmonic forms in the adiabatic limit, we prove an important (for our purposes) lemma that follows from the above statements.

Proposition 4.2.5. *Let $N = N(p)$ denote the page where the portion of the spectral sequence calculating $H^p(P)$ collapses. i.e. N is such that*

$$E_{N(p)-1}^p \neq E_{N(p)}^p = \cdots = E_\infty^p.$$

Suppose

$$\omega_\delta = v + \delta\omega_1 + \delta^2\omega_2 + \cdots$$

with

$$\omega_l \in E_\infty^\perp \quad (1 \leq l \leq L)$$

and

$$d_\delta\omega_\delta, d_\delta^*\omega_\delta \in \delta^{N+L}\Omega^*(P)[\delta].$$

Then, for $l \leq L$, the terms ω_l are unique.

Proof. Suppose $v + \delta\omega_1 + \cdots$ and $v + \delta\phi_1 + \cdots$ are such that $\omega_l, \phi_l \perp E_\infty^p$ for all

$l \geq 1$. Then, we have that

$$\begin{aligned}
& d_\delta(v + \delta\omega_1 + \delta^2\omega_2 + \cdots) - d_\delta(v + \delta\phi_1 + \delta^2\phi_2 + \cdots) \\
& \quad = d_\delta(\delta(\omega_1 - \phi_1) + \delta^2(\omega_2 - \phi_2) + \cdots) \in \delta^{N+L}\Omega^*(P)[\delta] \\
& d_\delta^*(v + \delta\omega_1 + \delta^2\omega_2 + \cdots) - d_\delta^*(v + \delta\phi_1 + \delta^2\phi_2 + \cdots) \\
& \quad = d_\delta^*(\delta(\omega_1 - \phi_1) + \delta^2(\omega_2 - \phi_2) + \cdots) \in \delta^{N+L}\Omega^*(P)[\delta]
\end{aligned}$$

Hence, $\omega_1 - \phi_1 \in E_{N+L-1}^p = E_\infty^p$, but we know that $\omega_1 - \phi_1 \in (E_\infty^p)^\perp$. Therefore, $\omega_1 = \phi_1$, and we continue inductively to show uniqueness of the higher order terms for $l \leq L$. \square

Theorem 4.2.6. (*Theorem 3.1 [For]*) Suppose $\omega \in E_\infty^{i,j}$. Then there is a unique formal power series

$$\omega_\delta = \omega + \delta\omega_1 + \delta^2\omega_2 + \cdots \in \Omega^{i+j}(P)[[\delta]]$$

such that

$$\omega_l \perp E_\infty \quad \forall l \geq 1,$$

and formally

$$d_\delta\omega_\delta = d_\delta^*\omega_\delta = 0.$$

Proof. The existence of a formally harmonic power series for $\omega \in E_\infty^{i,j}$ follows from the definition of E_∞ . The uniqueness of such a power series, assuming the higher order terms in E_∞^\perp , follows by Proposition 4.2.5 above. \square

These formal power series will be useful, so we give them a separate notation.

Definition 4.2.7.

$$\begin{aligned}
E_{\infty,\delta}^{i,j} & \stackrel{\text{def}}{=} \{\omega_\delta \in \Omega^*(P)[[\delta]] \mid \omega \in \Omega^{i,j}(P); \omega_l \in E_\infty^\perp (l \geq 1) \\
& \quad d_\delta\omega_\delta = d_\delta^*\omega_\delta = 0\}
\end{aligned}$$

Note that there is a bijective map

$$E_{\infty}^{i,j} \longleftrightarrow E_{\infty,\delta}^{i,j},$$

and the difference between the two \mathbb{R} -vector spaces is that $E_{\infty}^{i,j} \subset \Omega^{i,j}(P)$, while $E_{\infty,\delta}^{i,j} \subset \Omega^*(P)[[\delta]]$. We also note that

$$\text{Ker } L_{\delta}^k = \{\omega_{\delta} \in \Omega^k(P)[[\delta]] \mid d_{\delta}\omega_{\delta} = d_{\delta}^*\omega_{\delta} = 0\}.$$

Corollary 4.2.8. *The space $E_{\infty,\delta}^k[[\delta]] = E_{\infty,\delta}^k \otimes_{\mathbb{R}} \mathbb{R}[[\delta]]$, considered as a finite-dimensional $\mathbb{R}[[\delta]]$ -module, equals the space of formally L_{δ} -harmonic power series:*

$$E_{\infty,\delta}^k[[\delta]] = \text{Ker } L_{\delta}^k \subset \Omega^k(P)[[\delta]].$$

4.3 Relation to harmonic forms

To discuss the convergence of $\{E_K^p, \pi_K d_K \pi_K\}$ in relation to harmonic forms on P , we must impose further conditions. The “vertical” distribution B must be integrable, and the metric g_P on P must be “bundle-like.” This amounts to saying that (P, g_P) with distribution B locally has the structure of a Riemannian submersion (see (H1) in Section 5 of [For] for more specifics). In particular, if

$$F \hookrightarrow P \rightarrow M$$

is a fiber bundle of compact manifolds, with $P \rightarrow M$ a Riemannian submersion, then the metric and vertical distribution satisfy the necessary conditions. Also, if G is a compact Lie group with bi-invariant metric acting on (P, g_P) such that the orbits all have the same dimension, then the foliation of P associated to the orbits of G satisfies the hypotheses. The case of a principal bundle, which is our focus, is an example of both of these. All further discussion assumes these hypotheses are

satisfied.

We now wish to discuss the convergence of the spectral sequence in terms of the adiabatic limit of harmonic forms. Above, we treated δ as a formal variable. Now, we wish to also think of it as a number in $[0, 1]$ parameterizing the operators. To do so, we should be careful to distinguish between the analytically defined Laplacians (Δ_{g_δ} and L_{g_δ}) and the formal Laplacian L_δ . As a reminder,

$$\Delta_{g_\delta}^k : (\Omega^k(P), g_\delta) \rightarrow (\Omega^k(P), g_\delta) \quad \forall \delta \in (0, 1]$$

is the 1-parameter family of standard Hodge Laplacians with respect to the metric g_δ , and

$$L_{g_\delta}^k : (\Omega^k(P), g) \rightarrow (\Omega^k(P), g) \quad \forall \delta \in (0, 1]$$

is the 1-parameter family of rescaled Laplacian defined by (4.2.3)

$$\rho_\delta \Delta_{g_\delta}^k \rho_\delta^{-1} = L_{g_\delta}^k.$$

The formal operator on power series is denoted

$$L_\delta^k : \Omega^k(P)[[\delta]] \rightarrow \Omega^k(P)[[\delta]].$$

We then define the spaces $\text{Ker } \Delta_0^k$ and $\text{Ker } L_0^k$ by

$$\begin{aligned} \text{Ker } \Delta_0^k &= \lim_{\delta \rightarrow 0} \text{Ker } \Delta_{g_\delta}^k = \{\omega \in \Omega^k(P) \mid \exists \tilde{\omega} \in C^\infty([0, 1], \Omega^k(P)); \\ &\quad \tilde{\omega}(0) = \omega; \Delta_{g_\delta}^k \tilde{\omega}(\delta) = 0 \forall \delta > 0\} \end{aligned} \tag{4.3.1}$$

and

$$\begin{aligned} \text{Ker } L_0^k &= \lim_{\delta \rightarrow 0} \text{Ker } L_{g_\delta}^k = \{\omega \in \Omega^k(P) \mid \exists \tilde{\omega} \in C^\infty([0, 1], \Omega^k(P)); \\ &\quad \tilde{\omega}(0) = \omega; L_{g_\delta}^k \tilde{\omega}(\delta) = 0 \forall \delta > 0\} \end{aligned} \tag{4.3.2}$$

In other words, an element $\omega \in \text{Ker } L_0^k$ if and only if it is the limit of a 1-parameter

family of L_{g_δ} -harmonic forms. If $\tilde{\omega}$ is a smooth section of $\Omega^k(P) \rightarrow [0, 1]$ that is $L_{g_\delta}^k$ -harmonic for all $\delta > 0$, then it follows that the associated Taylor series at $\delta = 0$, which we denote $\omega_\delta \in \Omega^k(P)[[\delta]]$, is formally L_δ^k harmonic. In other words,

$$\tilde{\omega} \underset{\delta=0}{\sim} \omega_\delta = \omega + \delta\omega_1 + \delta^2\omega_2 + \cdots$$

and

$$L_\delta(\omega + \delta\omega_1 + \delta^2\omega_2 + \cdots) = 0.$$

This gives a map $\text{Ker } L_0^k \rightarrow E_\infty^k$. In fact, this map is an equality.

Theorem 4.3.1. *(Corollary 5.18 and Theorem 5.21 [For]) The L_{g_δ} -harmonic forms converge to E_∞ as $\delta \rightarrow 0$; i.e.*

$$\lim_{\delta \rightarrow 0} \text{Ker } L_{g_\delta}^k = \lim_{\delta \rightarrow 0} \rho_\delta \text{Ker } \Delta_\delta = E_\infty^k \subset \Omega^k(P).$$

In fact, the finite-dimensional vector spaces $\text{Ker } L_{g_\delta}$ extend smoothly to $\delta = 0$. More specifically, the spaces $\text{Ker } L_{g_\delta}^p$ form a C^∞ map from $[0, 1]$ to the space of $(\dim H^k(P))$ -dimensional subspaces of k -forms on P .

Corollary 4.3.2. *(Corollary 5.22 [For], Corollary 18 [MM]) The finite-dimensional vector spaces $\text{Ker } \Delta_{g_\delta}^k$ extend smoothly to $\delta = 0$.*

Therefore, in addition to the spaces $\text{Ker } L_0^k$ and $\text{Ker } \Delta_0^k$ being well-defined, the smoothness property allows us to discuss the 1-parameter families

$$C^\infty([0, 1], \text{Ker } L_{g_\delta}^k) \text{ and } C^\infty([0, 1], \text{Ker } \Delta_{g_\delta}^k).$$

This also gives the notion of the Taylor series at $\delta = 0$ of a family of harmonic forms.

To see the relationship of $\text{Ker } L_0$ with $\text{Ker } \Delta_0$, remember that for $\delta > 0$

$$\rho_\delta \Delta_{g_\delta} \rho_\delta^{-1} = L_{g_\delta}, \quad \text{Ker } \Delta_{g_\delta} = \rho_\delta^{-1} \text{Ker } L_{g_\delta}.$$

Since the isometry ρ_δ^{-1} involves dividing by powers of δ , we must be careful when applying at $\delta = 0$. To avoid these problems at $\delta = 0$, we will talk about the subspace of power series in $\Omega^*(P)[[\delta]]$ that are in the image of ρ_δ . In other words, it makes sense to apply ρ_δ^{-1} on them. We denote this $\rho_\delta \Omega^p(P)[[\delta]]$.

Proposition 4.3.3. *Suppose there is a power series of the form*

$$\omega + \delta\omega_1 + \cdots + \delta^k\omega_k + O(\delta^{k+1}) \in \text{Ker } L_\delta^k,$$

and ρ_δ^{-1} is defined on this power series. Then, applying ρ_δ^{-1} and taking the constant term gives an element of $\text{Ker } \Delta_0^k$:

$$\begin{aligned} & (\rho_\delta^{-1}(\omega + \delta\omega_1 + \cdots + \delta^k\omega_k))_{\delta=0} = \\ & = \omega^{0,k} + \omega_1^{1,k-1} + \omega_2^{2,k-2} + \cdots + \omega_k^{k,0} \in \text{Ker } \Delta_0^k \subset \Omega^k(P). \end{aligned}$$

Proof. Due to Theorem 4.3.1, there exists a family of L_{g_δ} -harmonic forms

$$\tilde{\omega} \in C^\infty([0, 1], \text{Ker } L_{g_\delta})$$

such that, close to $\delta = 0$,

$$\tilde{\omega}(\delta) \underset{\delta=0}{\sim} \omega + \delta\omega_1 + \cdots + \delta^k\omega_k + O(\delta^{k+1}).$$

Under the isometry ρ_δ^{-1} ,

$$\rho_\delta^{-1}\tilde{\omega} \in C^\infty([0, 1], \text{Ker } \Delta_{g_\delta}^k),$$

and

$$\rho_\delta^{-1}\tilde{\omega} \underset{\delta=0}{\sim} \rho_\delta^{-1}(\omega + \delta\omega_1 + \cdots + \delta^k\omega_k + O(\delta^{k+1})).$$

The isometry ρ_δ^{-1} divides at most by δ^k . Therefore,

$$\rho_\delta^{-1}(\omega + \delta\omega_1 + \cdots + \delta^k\omega_k + O(\delta^{k+1})) = (\omega^{0,k} + \omega_1^{1,k-1} + \omega_2^{2,k-2} + \cdots + \omega_k^{k,0}) + O(\delta).$$

Consequently,

$$\begin{aligned} (\rho_\delta^{-1}\tilde{\omega})(0) &= (\rho_\delta^{-1}(\omega + \delta\omega_1 + \cdots + \delta^k\omega_k))_{\delta=0} \\ &= \omega^{0,k} + \omega_1^{1,k-1} + \omega_2^{2,k-2} + \cdots + \omega_k^{k,0} \in \text{Ker } \Delta_0^k. \end{aligned}$$

□

Corollary 4.3.4. *There is an inclusion of vector spaces (as subspaces of $\Omega^*(P)$)*

$$E_\infty^{i,0} \subset \text{Ker } \Delta_0^i \subset \Omega^i(P).$$

Proof. Let $\omega \in E_\infty^{i,0}$. By definition, there exists a power series of the form

$$\omega + O(\delta) \in \text{Ker } L_\delta,$$

and consequently

$$\delta^k\omega + O(\delta^{k+1}) \in \text{Ker } L_\delta.$$

Applying Proposition 4.3.3, we see that

$$(\rho_\delta^{-1}(\delta^k\omega))_{\delta=0} = \omega \in \text{Ker } \Delta_0^k.$$

□

Proposition 4.3.5. *If $\omega_\delta \in \rho_\delta\Omega^k(P)[\delta]$ is a finite polynomial, and*

$$\omega_\delta \in \text{Ker } L_\delta,$$

then for any $\delta \geq 0$,

$$\rho_\delta^{-1}\omega_\delta \in \text{Ker } \Delta_{g_\delta}.$$

Proof. For any $\delta > 0$, $\omega_\delta \in \text{Ker } L_{g_\delta}$, since $L_\delta = L_{g_\delta}$ on polynomials. For all $\delta > 0$,

we have the equality

$$\rho_\delta^{-1} \text{Ker } L_{g_\delta} = \text{Ker } \Delta_{g_\delta},$$

and therefore $\rho_\delta^{-1}\omega_\delta \in \text{Ker } \Delta_{g_\delta}$ for $\delta > 0$. If ρ_δ^{-1} is defined on ω_δ at $\delta = 0$, then $(\rho_\delta^{-1}\omega_\delta)_{\delta=0} \in \text{Ker } \Delta_0$. \square

For us, the usefulness comes from combining Proposition 4.2.5 and Proposition 4.3.3, along with the isomorphism of $\{E_K\}$ with the Leray–Serre spectral sequence. First, we wish to calculate the adiabatic limit of L_{g_δ} -harmonic forms. Given a differential form, or a polynomial of forms $\omega_\delta \in \Omega^p(P)[\delta]$ (or 1-parameter family of forms), it is reasonable to calculate that d_δ and d_δ^* are 0 up to some power of δ . Thus, one can often show that

$$\lim_{\delta \rightarrow 0} L_{g_\delta} \omega_\delta = 0,$$

and even that it is 0 up some order of δ . A priori, that does not imply that $\omega_\delta(0)$ is the limit of some L_{g_δ} -harmonic form. But, the spectral sequence interpretation tells us that if it vanishes to a certain order, i.e.

$$\lim_{\delta \rightarrow 0} d_\delta \omega_\delta = 0(\delta^N), \quad \lim_{\delta \rightarrow 0} d_\delta^* \omega_\delta = 0(\delta^N),$$

then, higher order terms can be added so that it vanishes to *any* order of δ . Specifically, we only need to worry about making it harmonic up to an order determined by when the Leray–Serre spectral sequence collapses. This also applies to obtaining information about higher order terms in the Taylor series. These higher order terms can become important under the ρ_δ^{-1} isometry. Finally, Proposition 4.3.3 allows us to describe the adiabatic limit of Δ_{g_δ} harmonic forms in terms of the L_{g_δ} harmonic forms.

CHAPTER 5

HARMONIC FORMS ON A PRINCIPAL BUNDLE IN THE ADIABATIC LIMIT

Let (M, P, g, Θ) denote the setup of a principal G -bundle P with connection Θ over a Riemannian manifold (M, g) , as described in Chapter 3. Let g_G be *any* Ad -invariant metric on \mathfrak{g} , (or equivalently a bi-invariant metric on G). The spectral sequence machinery from Chapter 4 makes it possible to calculate the low-dimensional harmonic forms on P in the adiabatic limit. Given a topological assumption about P , namely that its first real characteristic class is 0, we then see the Chern–Simons 1- and 3-forms in the Hodge cohomology. In the following calculations, we use both the machinery of Chapter 4 and the complexes described in Chapter 3. An extra sheet with both complexes has been attached at the end of this document for the reader’s convenience. In the following order, we calculate:

- $\text{Ker } \Delta^1 \subset \Omega^1(P)$ when $G = U(n)$,
- $\text{Ker } \Delta^1 \subset \Omega^1(P)$ and $\text{Ker } \Delta_0^2 \subset \Omega^2(P)$ when G is semisimple,
- $\text{Ker } \Delta_0^3 \subset \Omega^3(P)$ when G is simple.

While the main goal is the calculation of $\text{Ker } \Delta_0^3$ for G simple, the other simpler cases are calculated in an analogous fashion, providing a good warm-up for the main proof.

5.1 1-forms when $G = U(n)$

We now examine the case where $G = U(n)$ ($n \geq 1$). Assume we have (M, P, g, Θ) , and let g_G be *any* Ad -invariant metric on \mathfrak{g} . This produces the 1-parameter family of metrics g_δ as described in Chapter 3.

The part of the Leray–Serre spectral sequence which calculates the 1-dimensional real cohomology is

$$\begin{array}{ccccc}
 & & H^1(U(n); \mathbb{R}) & \xrightarrow{\quad \cdots \quad} & \\
 & & & \searrow^{d_2} & \\
 & H^0(M; \mathbb{R}) & & H^1(M; \mathbb{R}) & \longrightarrow & H^2(M; \mathbb{R})
 \end{array}$$

The space $H^1(U(n); \mathbb{R})$ is canonically isomorphic to \mathbb{R} , and the generator is sent to $c_1(P)$ by the d_2 differential. If $c_1(P) = 0 \in H^2(M; \mathbb{R})$, then the d_2 differential is 0, and the portion of spectral sequence calculating $H^1(P; \mathbb{R})$ collapses at $N = N(1) = 2$. Therefore,

$$E_\infty^{0,1} \cong H^1(U(n); \mathbb{R}) \cong \mathbb{R}, \quad E_\infty^{1,0} \cong H^1(M; \mathbb{R}). \quad (5.1.1)$$

If $c_1(P) \neq 0$, then d_2 is injective on $H^1(U(n); \mathbb{R})$. Therefore,

$$E_\infty^{0,1} = \text{Ker } d_2 (H^1(U(n); \mathbb{R})) = 0, \quad E_\infty^{1,0} = E_2^{1,0} \cong H^1(M; \mathbb{R}). \quad (5.1.2)$$

Remark 5.1.1. *In this situation, we can calculate the harmonic forms explicitly, making the spectral sequence machinery unnecessary. However, we still use the interpretation of d, d^* in terms of the bi-graded complexes (3.3.2) and (3.3.3).*

In the following, $\frac{i}{2\pi} \text{Tr}(\Theta) \in \Omega^1(P)$ is the Chern–Simons 1-form for a unitary bundle (Definition 2.5.2), and $\mathcal{H}_g^1(M)$ is the finite-dimensional vector space

of harmonic forms on (M, g) . As noted in Proposition 2.5.3, the Chern–Simons 1-form $\frac{i}{2\pi}Tr(\Theta)$ is right-invariant. Also, the form $\frac{i}{2\pi}Tr(\Theta)$ lives in the $(0, 1)$ component, since $T^H P = \text{Ker } \Theta$. We will not distinguish notationally between the form $\frac{i}{2\pi}Tr(\Theta) \in \Omega^{0,1}(P)^G$ and its image in $\Omega^0(M; \mathfrak{g}_P^*)$ under the isomorphism of complexes in 3.1.7

$$\{\Omega^{i,j}(P)^G\} \xleftarrow{\pi^*} \{\Omega^i(M; \Lambda^j \mathfrak{g}_P^*)\}.$$

Also, for any subspace $W \subset \Omega^*(P)$, we call $W \subset \Omega^*(P)[[\delta]]$ the image of W under the inclusion

$$\Omega^*(P) \hookrightarrow \Omega^*(P)[[\delta]]$$

(where $\Omega^*(P)$ includes as constant power series.)

Proposition 5.1.2. *Given (M, P, g, Θ) with $G = U(n)$ ($n \geq 1$), if $c_1(P) = 0 \in H^2(M; \mathbb{R})$, then*

$$E_{\infty, \delta}^1 = \left(\mathbb{R} \left[\frac{i}{2\pi} Tr(\Theta) - \delta \pi^* h \right] \oplus \pi^* \mathcal{H}_g^1(M) \right) \subset \Omega^1(P)[[\delta]],$$

where $h \in \Omega^1(M)$ is the unique form such that

$$dh = c_1(P, \Theta), \quad h \in d^* \Omega^2(M).$$

Proof. The proof follows in two parts. First, we show that for any element $\beta \in \mathcal{H}_g^1(M)$, $\pi^* \beta \in E_{\infty, \delta}^{1,0}$. Second, we show that $\frac{i}{2\pi} Tr(\Theta) - \delta \pi^* h \in E_{\infty, \delta}^{0,1}$. As a reminder, $E_{\infty, \delta}^{i,j} \subset \Omega^{i+j}(P)[[\delta]]$ was defined in Definition 4.2.7 by

$$E_{\infty, \delta}^{i,j} \stackrel{\text{def}}{=} \{ \omega_\delta \in \Omega^*(P)[[\delta]] \mid \omega \in \Omega^{i,j}(P); \omega_l \in E_\infty^\perp (l \geq 1) \}$$

$$d_\delta \omega_\delta = d_\delta^* \omega_\delta = 0\}.$$

$$\underline{\pi^* \mathcal{H}_g^1(M)} = E_{\infty, \delta}^{1,0}$$

Let $\beta \in \mathcal{H}_g^1(M)$. Then, $\pi^*\beta \in \Omega^{1,0}(P)^G$ is a right-invariant horizontal form on P . We proceed to show that $d_\delta(\pi^*\beta) = d_\delta^*(\pi^*\beta) = 0$. Since $\pi^*\beta$ is right-invariant, we calculate these using the isomorphic complexes (3.3.2) and (3.3.3). For an element of $\Omega^{1,0}(P)^G = \pi^*\Omega^1(M; \mathbb{R})$, $d_\delta(\pi^*\bullet) = \pi^*(d_M\bullet)$. Since h is closed,

$$d_\delta(\pi^*h) = \pi^*(d_M h) = 0.$$

Likewise, since h is also coclosed,

$$d_\delta^*(\pi^*h) = \pi^*(d_M^* h) = 0.$$

Note that the $(2, -1^*)$ component of d_δ^* is 0 here for dimensional reasons. We have now shown that for any element $\beta \in \pi^*\mathcal{H}_g^1(M)$, $\pi^*\beta \in \Omega^{1,0}(P)$ and $d_\delta(\pi^*\beta) = d_\delta^*(\pi^*\beta) = 0$. Since there are no higher order terms (with respect to δ) in the power series $\pi^*\beta$, we have satisfied all the conditions for $\pi^*\beta \in E_{\infty, \delta}^{1,0}$. Therefore,

$$\pi^*\mathcal{H}_g^1(M) \subset E_{\infty, \delta}^{1,0}.$$

By the isomorphism of the filtration $\{E_K^{i,j}\}$ with the Leray–Serre spectral sequence, along with the description of $E_\infty^{1,0}$ in equation (5.1.1), we see

$$\dim E_{\infty, \delta}^{1,0} = \dim E_\infty^{1,0} = \dim H^1(M; \mathbb{R}).$$

Therefore, our inclusion of vector spaces is surjective:

$$\pi^*\mathcal{H}_g^1(M) = E_{\infty, \delta}^{1,0}.$$

As a side remark, the relationship $\pi^*\mathcal{H}_g^i(M) = E_2^{i,0}$ holds true in general and is proved in Proposition 5.2.1. The fact that $E_2^{1,0} = E_\infty^{1,0}$ gives us the constant term in $E_{\infty, \delta}^{1,0}$.

$$\mathbb{R}[\frac{i}{2\pi}Tr(\Theta) - \delta\pi^*h] = E_{\infty,\delta}^{0,1}$$

First, the existence of an $h \in \Omega^1(M)$ such that $dh = c_1(P, \Theta)$ is given by the fact that as a de Rham class,

$$[c_1(P, \Theta)] = c_1(P) = 0 \in H^2(M; \mathbb{R}).$$

Therefore, $c_1(P, \Theta) \in \Omega^2(M)$ is an exact form. Using the Hodge decomposition of $\Omega^1(M)$, there exists a unique form h such that $dh = c_1(P, \Theta)$, and $h \in d^*\Omega^2(M)$ (Corollary 2.4.2).

We proceed by directly computing d_δ and d_δ^* of $\frac{i}{2\pi}Tr(\Theta) - \pi^*h$. Because this is a right-invariant form on P , we can calculate using the isomorphic complexes (3.3.2) and (3.3.3). Under the bi-grading, $\frac{i}{2\pi}Tr(\Theta) \in \Omega^{0,1}(P)^G$ and $\pi^*h \in \Omega^{1,0}(P)^G$. Furthermore, the derivative of the Chern–Simons 1-form is the first Chern form (Proposition 2.5.3), so

$$d(\frac{i}{2\pi}Tr(\Theta) - \pi^*h) = \pi^*c_1(P, \Theta) - \pi^*c_1(P, \Theta) = 0 \in \Omega^{2,0}(P)^G.$$

The isometry ρ_δ is given by multiplying by a power of δ , with that power determined by the number of horizontal components in a form:

$$\begin{aligned} \rho_\delta : \Omega^{i,j}(P)[|\delta|] &\longrightarrow \Omega^{i,j}(P)[|\delta|] \\ \psi &\longmapsto \delta^i \psi \end{aligned}$$

The rescaled derivative d_δ is given by $d_\delta = \rho_\delta d \rho_\delta^{-1}$. Therefore, if ψ is d -closed, then $\rho_\delta \psi$ is d_δ -closed. Consequently,

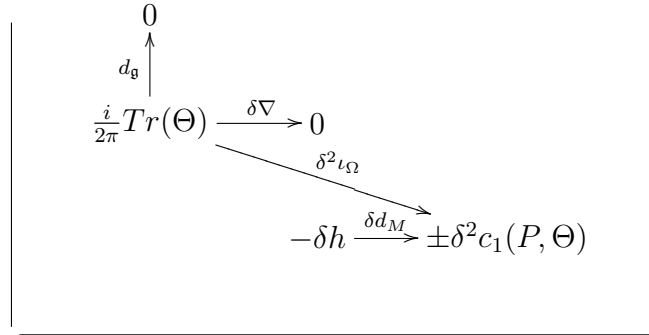
$$d_\delta \rho_\delta \left(\frac{i}{2\pi}Tr(\Theta) - \pi^*h \right) = d_\delta \left(\frac{i}{2\pi}Tr(\Theta) - \delta\pi^*h \right) = 0.$$

Alternatively, one could look at how the components of d_δ act in the complex (3.3.2). The isomorphism $\{\Omega^i(M; \Lambda^j \mathfrak{g}_P^*)\} \rightarrow \{\Omega^{i,j}(P)^G\}$ is denoted π^* . We then explicitly

see

$$\begin{aligned} d_\delta \left(\frac{i}{2\pi} \text{Tr}(\Theta) - \delta\pi^*h \right) &= \pi^* \left(d_{\mathfrak{g}} \left(\frac{i}{2\pi} \text{Tr}(\Theta) \right) + \delta d_\nabla \left(\frac{i}{2\pi} \text{Tr}(\Theta) \right) + \delta^2 \left(\iota_\Omega \frac{i}{2\pi} \text{Tr}(\Theta) - d_M h \right) \right) \\ &= \pi^* (0 + 0 + \delta^2 (c_1(P, \Theta) - c_1(P, \Theta))) = 0. \end{aligned}$$

While this issue of the bi-grading may seem a bit confusing, one can visualize the calculation in the picture below. In this, we are thinking of the forms living in the complex (3.3.2), and the arrows represent the (a priori) non-zero components of d_δ . Finally, we denote the fact that $\delta^2 \iota_\Omega \frac{i}{2\pi} \text{Tr}(\Theta) = -\delta^2 d_M h$ by the \pm sign in the image. In other words, these two elements cancel each other. An extra page with this complex is attached at the end of the thesis. While this may all seem a bit unnecessary in this calculation, the author finds these pictures extremely helpful in organizing the calculations in the upcoming Proposition 5.3.1.



We now calculate $d_\delta^* \left(\frac{i}{2\pi} \text{Tr}(\Theta) - \delta\pi^*h \right)$. Since both forms are right-invariant, when viewed in the complex (3.3.3) the only non-trivial components of $d_\delta^* \left(\frac{i}{2\pi} \text{Tr}(\Theta) - \delta\pi^*h \right)$ are given by

$$d_{\mathfrak{g}}^* \left(\frac{i}{2\pi} \text{Tr}(\Theta) \right) \text{ and } d_M^*(h),$$

or the $d^{0,1*}$ and $d^{1,0*}$, respectively. Since $i_x^* \frac{i}{2\pi} \text{Tr}(\Theta) \in \mathfrak{g}_P^*$ (the restriction of $\frac{i}{2\pi} \text{Tr}(\Theta) \in \Omega^0(M; \mathfrak{g}_P^*)$ to a fiber) is Ad -invariant (Proposition 2.5.3), and Ad -invariant elements of \mathfrak{g}_P^* are harmonic (Proposition 2.4.4), then $\frac{i}{2\pi} \text{Tr}(\Theta)$ is harmonic when restricted

to the fibers. Therefore,

$$d_{\mathfrak{g}}^*\left(\frac{i}{2\pi}Tr(\Theta)\right) = 0.$$

By definition, $h \in d^*\Omega^2(M)$, and therefore

$$d_M^*h \in (d_M^*)^2\Omega^2(M; \mathbb{R}) = 0.$$

Hence,

$$d_{\delta}^*\left(\frac{i}{2\pi}Tr(\Theta) - \delta\pi^*h\right) = \pi^*\left(d_{\mathfrak{g}}^*\frac{i}{2\pi}Tr(\Theta) - \delta^2d_M^*h\right) = 0$$

As above, we can draw this in our complex (3.3.3). This looks like

$$\begin{array}{c} \left. \begin{array}{c} \frac{i}{2\pi}Tr(\Theta) \\ \downarrow 0 \\ 0 \end{array} \right\} \begin{array}{c} \leftarrow \delta d_M^* \\ \leftarrow -\delta h \end{array} \end{array}$$

In particular, the picture makes it easy to identify the potentially non-trivial components of d_{δ}^* . It should be noted that we do not have to worry about $d^{2,-1*}h$ for dimensional reasons. This does not hold true in the situations for higher-dimensional forms.

We have now shown that $d_{\delta}\left(\frac{i}{2\pi}Tr(\Theta) - \delta\pi^*h\right) = d_{\delta}^*\left(\frac{i}{2\pi}Tr(\Theta) - \delta\pi^*h\right) = 0$, and the constant term $\frac{i}{2\pi}Tr(\Theta) \in \Omega^{0,1}(P)$. In order for $\frac{i}{2\pi}Tr(\Theta) - \delta\pi^*h$ to be in $E_{\infty,\delta}^{0,1}$, all higher order (with respect to δ) terms must be in E_{∞}^{\perp} . In the above discussion, we showed that

$$E_{\infty}^{1,0} = \pi^*\mathcal{H}_g^1(M).$$

Our assumption that $h \in d^*\Omega^2(M)$ implies that

$$h \perp \mathcal{H}_g^1(M).$$

This is the only higher-order term, and so we have now verified that

$$\frac{i}{2\pi}Tr(\Theta) - \delta\pi^*h \in E_{\infty,\delta}^1.$$

By the isomorphism of our filtration on forms $\{E_K^{i,j}\}$ with the Leray–Serre spectral sequence and (5.1.1),

$$\dim E_{\infty}^{0,1} = \dim E_{\infty,\delta}^{0,1} = 1.$$

Consequently, the 1-dimensional vector space spanned by $\frac{i}{2\pi}Tr(\Theta) - \delta\pi^*h$ is equal to $E_{\infty,\delta}^{0,1}$, giving us

$$\mathbb{R}[\frac{i}{2\pi}Tr(\Theta) - \delta\pi^*h] = E_{\infty,\delta}^{0,1}.$$

Finally, $E_{\infty,\delta}^1 = E_{\infty,\delta}^{0,1} \oplus E_{\infty,\delta}^{1,0}$, giving us the desired result

$$E_{\infty,\delta}^1 = \mathbb{R}[\frac{i}{2\pi}Tr(\Theta) - \pi^*h] \oplus \pi^*\mathcal{H}_g^1(M).$$

□

Theorem 5.1.3. *Given (M, P, g, Θ) with $G = U(n)$ ($n \geq 1$), if $c_1(P) = 0 \in H^2(M; \mathbb{R})$, then for any $\delta \geq 0$*

$$\text{Ker } \Delta_{g_\delta}^1 = \mathbb{R}[\frac{i}{2\pi}Tr(\Theta) - \pi^*h] \oplus \pi^*\mathcal{H}_g^1(M) \subset \Omega^1(P)$$

where $h \in \Omega^1(M)$ is the unique form such that

$$dh = c_1(P, \Theta), \quad h \in d^*\Omega^2(M).$$

Proof. The proof follows almost directly from Propositions 4.3.5 and 5.1.2. To make things a bit more explicit, we go ahead and show this directly.

In general, $\text{Ker } L_\delta^k$ is spanned by $E_{\infty,\delta}^k$ under multiplication by elements of $\mathbb{R}[|\delta|]$ (Corollary 4.2.8); i.e.

$$\text{Ker } L_\delta^k = E_{\infty,\delta}^k[|\delta|] = E_{\infty,\delta}^k \otimes_{\mathbb{R}} \mathbb{R}[|\delta|].$$

Proposition 5.1.2 says that under our given assumptions,

$$\text{Ker } L_\delta^1 = E_{\infty, \delta}^1[|\delta|] = \left(\mathbb{R} \left[\frac{i}{2\pi} \text{Tr}(\Theta) - \delta \pi^* h \right] \oplus \pi^* \mathcal{H}_g^1(M) \right) [|\delta|].$$

These power series are actually polynomials, and the formal L_δ coincides with the analytically defined L_{g_δ} on polynomials. Therefore, for any $\delta \geq 0$,

$$\text{Ker } L_{g_\delta}^1 = \mathbb{R} \left[\frac{i}{2\pi} \text{Tr}(\Theta) - \delta \pi^* h \right] \oplus \pi^* \mathcal{H}_g^1(M).$$

The isometry

$$\begin{aligned} \rho_\delta : (\Omega^{i,j}(P), g_\delta) &\rightarrow (\Omega^{i,j}(P), g) \\ \psi &\mapsto \delta^i \psi \end{aligned}$$

relates the two Laplacians L_{g_δ} and Δ_{g_δ} via (4.2.3)

$$L_{g_\delta} = \rho_\delta \Delta_{g_\delta} \rho_\delta^{-1}.$$

Also, letting ρ_δ act formally on power series, we have the formal relationship (ignoring the issue of where ρ_δ^{-1} is defined)

$$L_\delta = \rho_\delta \Delta_\delta \rho_\delta^{-1}.$$

Therefore, ρ_δ^{-1} takes elements in $\text{Ker } L_{g_\delta}$ to elements in $\text{Ker } \Delta_{g_\delta}$ for any $\delta > 0$. When defined on a power series or polynomial, then ρ_δ^{-1} maps $\text{Ker } L_\delta$ to $\text{Ker } \Delta_\delta$. Therefore, for any δ ,

$$\rho_\delta^{-1} \left(\frac{i}{2\pi} \text{Tr}(\Theta) - \delta \pi^* h \right) = \frac{i}{2\pi} \text{Tr}(\Theta) - \pi^* h \in \text{Ker } \Delta_{g_\delta}$$

and

$$\rho_\delta^{-1} (\delta \pi^* \mathcal{H}_g^1(M)) = \pi^* \mathcal{H}_g^1(M) \subset \text{Ker } \Delta_{g_\delta}.$$

For any $\delta \geq 0$, $\text{Ker } L_{g_\delta} \cong \text{Ker } \Delta_{g_\delta}$, so the above maps give an inclusion of vector

spaces which is therefore an isomorphism. We conclude that for any $\delta \geq 0$,

$$\text{Ker } \Delta_{g_\delta}^1 = \left(\mathbb{R} \left[\frac{i}{2\pi} \text{Tr}(\Theta) - \pi^* h \right] \oplus \pi^* \mathcal{H}_g^1(M) \right).$$

□

Remark 5.1.4. *There are no higher order powers of δ in the basis elements of $\text{Ker } \Delta_{g_\delta}$. In fact, there is a canonical connection on the bundle $\text{Ker } \Delta_{g_\delta}^1 \rightarrow [0, 1]$ given by the isomorphism of $\text{Ker } \Delta_{g_\delta}^1 \cong H^1(P; \mathbb{R})$. In this situation, since the basis elements of $\text{Ker } \Delta_{g_\delta}^1$ have no higher order powers of δ , the bundle $\text{Ker } \Delta_{g_\delta}^1 \cong \text{Ker } \Delta_0^1 \times [0, 1]$, and the connection is the trivial connection. In other words, the harmonic representative of a class in $H^1(P; \mathbb{R})$ is independent of the scaling on fibers.*

Theorem 5.1.5. *Given (M, P, g, Θ) with $G = U(n)$ ($n \geq 1$) and $c_1(P) \neq 0 \in H^2(M; \mathbb{R})$, then for any $\delta \geq 0$*

$$\text{Ker } \Delta_{g_\delta}^1 = \pi^* \mathcal{H}_g^1(M).$$

Proof. This follows from the description of E_∞^1 in (5.1.2), along with the exact same proof for

$$\pi^* \mathcal{H}_g^1(M) \subset \text{Ker } \Delta_{g_\delta}^1$$

given in the previous Theorem 5.1.3. The fact that $c_1(P) \neq 0$ implies that we do not have to describe the cohomology class it generates. □

5.2 1- and 2-forms when G is semisimple

If G is a compact semi-simple Lie group, then $H^1(G; \mathbb{R}) = 0$, and $H^2(G; \mathbb{R}) = 0$ as well. The E_2 page of the Leray–Serre spectral sequence is isomorphic to

$$E_2^{i,j} \cong H^i(M; H^j(G; \mathbb{R})).$$

Consequently, the portion of the E_2 page which calculates $H^1(P; \mathbb{R})$ and $H^2(P; \mathbb{R})$ looks like

$$\begin{array}{cccc}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 H^0(X; \mathbb{R}) & H^1(X; \mathbb{R}) & H^2(X; \mathbb{R}) & H^3(X; \mathbb{R})
 \end{array}
 \tag{5.2.1}$$

Since there can be no non-trivial differentials in this portion, we see that the terms where the sequence collapses in calculating $H^1(P; \mathbb{R})$ and $H^2(P; \mathbb{R})$ are

$$N = N(1) = N(2) = 2.$$

Consequently,

$$E_\infty^1 = E_2^{1,0} \cong H^1(M; \mathbb{R}), \quad E_\infty^2 = E_2^{2,0} \cong H^2(M; \mathbb{R}). \tag{5.2.2}$$

Before proceeding, we first, we point out a useful and general property.

Proposition 5.2.1. *Given (M, P, g, Θ) , then for any i ,*

$$E_2^{i,0} = \pi^* \mathcal{H}_g^i(M).$$

Proof. By Definition 4.2.2,

$$\begin{aligned}
 E_2^{i,0} &= \{ \psi \in \Omega^{i,0}(P) \text{ such that } \exists \psi_1, \dots \\
 &\quad d_\delta(\psi + \delta\psi_1 + \dots) \in \delta^2 \Omega^*(P)[\delta] \\
 &\quad d_\delta^*(\psi + \delta\psi_1 + \dots) \in \delta^2 \Omega^*(P)[\delta] \}
 \end{aligned}$$

For $\beta \in \mathcal{H}_g^i(M)$, then $\pi^*\beta \in \Omega^{i,0}(P)^G$, and the only non-zero component of d_δ is $d^{1,0}$. Under the isomorphism with (3.3.2), we see that because β is closed

$$d_\delta(\pi^*\beta) = \delta\pi^*(d_M\beta) = 0.$$

Drawn in the complex (3.3.2), this looks like

$$\begin{array}{ccc} & & \uparrow \\ & & 0 \\ & & \pi^*\beta \xrightarrow{\delta d_M} 0 \end{array}$$

Calculating $d_\delta^*(\pi^*h)$ using the isomorphic complex (3.3.3), we see that because β is coclosed,

$$d_\delta^*(\pi^*\beta) = \pi^*(\delta d_M^*\beta + \delta^2\iota_\Omega^*\beta) = \delta^2\pi^*(\iota_\Omega^*\beta).$$

Therefore, $d_\delta^*(\pi^*\beta) \in \delta^2\Omega^*(P)[\delta]$. Drawn in the complex (3.3.3), this looks like

$$\begin{array}{ccc} & & \delta^2\iota_\Omega^* \\ & & \swarrow \\ & & 0 \xleftarrow{\delta d_M^*} \pi^*\beta \end{array}$$

We draw the $\delta^2\iota_\Omega^*$ as dotted since we do not need to worry about it.

We have shown that any element $\beta \in \mathcal{H}_g^i(M)$ satisfies

$$d_\delta(\pi^*\beta) = 0; \quad d_\delta^*(\pi^*\beta) \in \delta^2\Omega^*(P)[\delta].$$

This gives an inclusion

$$\pi^*\mathcal{H}_g^i(M) \subset E_2^{i,0}. \quad (5.2.3)$$

By the isomorphism of $\{E_K^{i,j}\}$ with the Leray–Serre spectral sequence,

$$\dim E_2^{i,0} = \dim H^i(M; \mathbb{R}).$$

Therefore, the inclusion (5.2.3) is an equality. \square

Proposition 5.2.2. *Given (M, P, g, Θ) with G semisimple, then*

$$E_{\infty, \delta}^1 = \pi^* \mathcal{H}_g^1(M) \subset \Omega^1(P)[[\delta]].$$

Proof. The Leray–Serre spectral sequence calculating $H^1(P; \mathbb{R})$ collapses at $N = N(1) = 2$, as displayed in (5.2.1). The isomorphism between the Leray–Serre spectral sequence and our adiabatic spectral sequence gives

$$E_{\infty}^1 = E_2^{1,0} = \pi^* \mathcal{H}_g^1(M),$$

with the second equality given by Proposition 5.2.1. We now show that any element $\beta \in \mathcal{H}_g^1(M)$ satisfies

$$d_{\delta}(\pi^* \beta) = d_{\delta}^*(\pi^* \beta) = 0.$$

This was actually proven in Proposition 5.1.2, and we give the same argument here.

To calculate $d_{\delta}(\pi^* \beta)$, we see via (3.3.2)

$$d_{\delta}(\pi^* h) = \pi^*(\delta d_M h) = 0.$$

Furthermore, via (3.3.3),

$$d_{\delta}^*(\pi^* h) = \pi^*(\delta d_M^* h) = 0.$$

There is no $\delta^2 \iota_{\Omega}^* \beta$ term for dimensional reasons. Therefore, we have the stronger statement that the constant power series $\pi^* \beta$ is, in fact, the unique power series

associated to $\pi^*\beta \in E_\infty^{1,0}$ via Theorem 4.2.6, and

$$E_{\infty,\delta}^{1,0} = \pi^*\mathcal{H}_g^1(M).$$

□

Theorem 5.2.3. *Given (M, P, g, Θ) with G semisimple, then for any δ*

$$\text{Ker } \Delta_{g\delta}^1 = \pi^*\mathcal{H}_g^1(M) \subset \Omega^1(P).$$

Proof. By Proposition 5.2.2 and Corollary 4.2.8, we have

$$\text{Ker } L_\delta^1 = E_{\infty,\delta}^1[[\delta]] = \pi^*\mathcal{H}_g^1(M)[[\delta]].$$

Since ρ_δ^{-1} is defined on $\delta\mathcal{H}_g^1(M)$, Proposition 4.3.5 implies that for any $\delta \geq 0$,

$$\rho_\delta^{-1}(\delta\pi^*\mathcal{H}_g^1(M)) = \pi^*\mathcal{H}_g^1(M) \subset \text{Ker } \Delta_{g\delta}^1.$$

Since this map $\pi^*\mathcal{H}_g^1(M) \hookrightarrow \text{Ker } \Delta_{g\delta}^1$ is injective for all $\delta \geq 0$, and $\text{Ker } L_{g\delta}^1 \cong \text{Ker } \Delta_{g\delta}^1$, we have an equality of vector spaces

$$\pi^*\mathcal{H}_g^1(M) = \text{Ker } \Delta_{g\delta}^1 \subset \Omega^1(P).$$

□

Proposition 5.2.4. *Given (M, P, g, Θ) with G semisimple, then*

$$E_\infty^2 = E_\infty^{2,0} = \pi^*\mathcal{H}_g^2(M) \subset \Omega^2(P).$$

Proof. The Leray–Serre spectral sequence calculating $H^2(P; \mathbb{R})$ collapses at $N = N(2) = 2$, as displayed in (5.2.1). This fact, combined with the description of E_2 given in Proposition 5.2.1, implies that

$$E_\infty^2 = E_2^{2,0} = \pi^*\mathcal{H}_g^2(M).$$

□

Theorem 5.2.5. *Given (M, P, g, Θ) with G semisimple, then*

$$\text{Ker } \Delta_0^2 = \lim_{\delta \rightarrow 0} \text{Ker } \Delta_{g\delta}^2 = \pi^* \mathcal{H}_g^2(M) \subset \Omega^2(P).$$

Proof. Proposition 5.2.4 says that $E_\infty^{2,0} = \pi^* \mathcal{H}_g^2(M)$. Corollary 4.3.4 states that the purely horizontal part of E_∞ includes into $\text{Ker } \Delta_0$. In other words,

$$E_\infty^{2,0} \subset \text{Ker } \Delta_0^2.$$

Since $E_\infty^2 = E_\infty^{2,0}$,

$$\pi^* \mathcal{H}_g^2(M) = \text{Ker } \Delta_0^2.$$

□

5.3 3-forms when G is simple

We now examine the case where G is a compact simple Lie group. We denote the Chern–Simons 3-form (Definition 2.5.5)

$$\alpha(\Theta) = \langle \Theta \wedge \Omega \rangle - \frac{1}{6} \langle \Theta \wedge [\Theta \wedge \Theta] \rangle \in \Omega^3(P).$$

Under the bi-grading, we write

$$\alpha(\Theta) = \alpha^{0,3} + \alpha^{2,1},$$

where

$$\alpha^{0,3} = -\frac{1}{6} \langle \Theta \wedge [\Theta \wedge \Theta] \rangle \in \Omega^{0,3}(P)$$

and

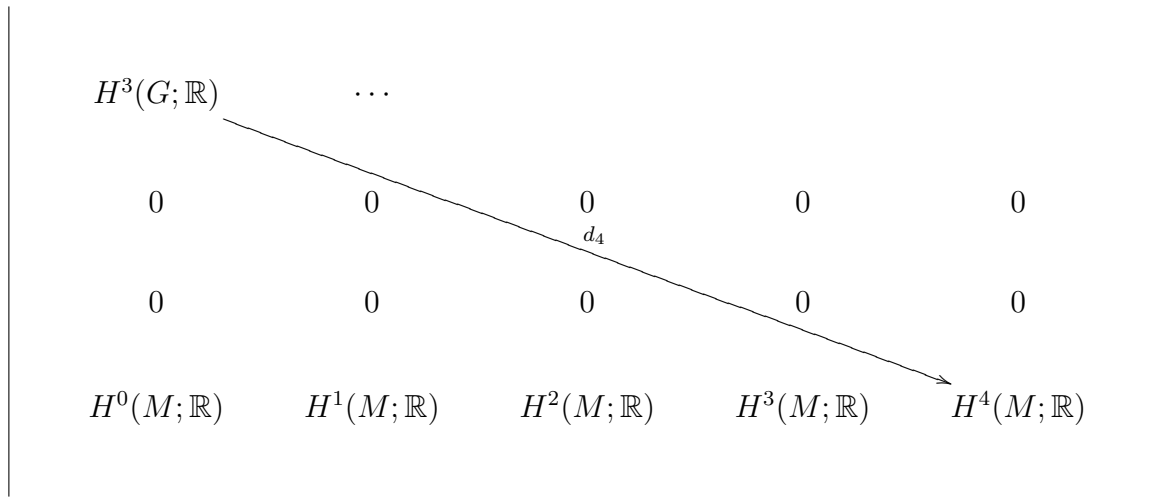
$$\alpha^{2,1} = \langle \Theta \wedge \Omega \rangle \in \Omega^{2,1}(P).$$

This bi-grading follows from the definition of $T^H P = \text{Ker } \Theta$ (2.3.2), and the fact that the curvature $\Omega \in \Omega^2(M; \mathfrak{g}_P)$ (Proposition 2.3.5). The corresponding 4-dimensional Chern–Weil form is denoted

$$\langle \Omega \wedge \Omega \rangle \in \Omega^4(M).$$

The proofs use $H^1(G; \mathbb{R}) = H^2(G; \mathbb{R}) = 0$, and the condition that G is simple implies $H^3(G; \mathbb{R}) \cong \mathbb{R}$. We then only have to worry about the vanishing of a single characteristic class.

We first describe the portion of the Leray–Serre spectral sequence which calculates $H^3(P; \mathbb{R})$ (the universal case was discussed in Example 4.1.1). Due to the fact that $H^1(G; \mathbb{R}) = H^2(G; \mathbb{R}) = 0$, the only non-trivial differential is on the E_4 page. The picture shows $E_2 = E_3 = E_4$ and the non-trivial d_4 .



$H^3(G; \mathbb{R})$ is isomorphic to \mathbb{R} , and the (standard) generator is sent to $[\langle \Omega \wedge \Omega \rangle] \in H^4(M; \mathbb{R})$ by d_4 (see Example 4.1.1).

The condition $[\langle \Omega \wedge \Omega \rangle] = 0 \in H^4(M; \mathbb{R})$ is equivalent to the above spectral

sequence collapsing at $N = N(3) = 2$. This also implies that

$$E_\infty^3 = E_2^3 = E_2^{0,3} \oplus E_2^{3,0} \cong H^3(G; \mathbb{R}) \oplus H^3(M; \mathbb{R}). \quad (5.3.1)$$

If $[\langle \Omega \wedge \Omega \rangle] \neq 0 \in H^4(M; \mathbb{R})$, then we see that d_4 is injective on the $(0, 3)$ component, and hence

$$E_\infty^3 = E_\infty^{3,0} = E_2^{3,0} \cong H^3(M; \mathbb{R}). \quad (5.3.2)$$

Proposition 5.3.1. *Given (M, P, g, Θ) with G simple and $[\langle \Omega \wedge \Omega \rangle] = 0 \in H^4(M; \mathbb{R})$, then*

$$E_\infty^3 = \mathbb{R}[\alpha^{0,3}] \oplus \mathcal{H}_g^3(M) \subset \Omega^3(P).$$

Furthermore, the unique power series in $E_{\infty, \delta}^3$ given by $\alpha^{0,3}$ is of the form

$$\alpha^{0,3} + \delta^2 \alpha^{2,1} - \delta^3 \pi^* h + O(\delta^3) \in \Omega^3(P)[[\delta]],$$

where $h \in \Omega^3(M)$ is the unique form such that

$$dh = \langle \Omega \wedge \Omega \rangle, \quad h \in d^* \Omega^4(M).$$

Proof. The order of proof is

- Show $E_\infty^{3,0} = \pi^* \mathcal{H}_g^3(M)$.
- First attempt at showing exists formal power series of form $\alpha^{0,3} + \delta^2 \alpha^{1,2} + O(\delta^3) \in E_{\infty, \delta}^{0,3}$.
- Show there exists formal power series of the form $\alpha^{0,3} + \delta^2 \alpha^{1,2} + O(\delta^3) \in E_{\infty, \delta}^{0,3}$.
- Show that in the above power series, the $(3, 0)$ component of the δ^3 term is h .

$$\underline{E_\infty^{3,0} = \pi^* \mathcal{H}_g^3(M)}$$

The description of $E_\infty^{3,0}$ follows immediately from the fact that the spectral sequence

for $H^3(P; \mathbb{R})$ collapses at $N = N(3) = 2$, along with the description of $E_2^{3,0}$ in Proposition 5.2.1;

$$E_\infty^{3,0} = E_2^{3,0} = \pi^* \mathcal{H}_g^3(M).$$

Now we move on to the main part of this Proposition. Since $N = N(3) = 2$, to show that there exists a formal power series of the form

$$\alpha^{0,3} + \delta^2 \alpha^{2,1} + O(\delta^3) \in E_{\infty, \delta}^{0,3},$$

it suffices to show, by Proposition 4.2.5, that there exist forms $h, \psi^{1,2}$, such that

$$\begin{aligned} d_\delta (\alpha^{0,3} + \delta^2 \alpha^{2,1} - \delta^3 (\pi^* h + \psi^{1,2})) &\in \delta^4 \Omega^4(P)[\delta] \\ d_\delta^* (\alpha^{0,3} + \delta^2 \alpha^{2,1} - \delta^3 (\pi^* h + \psi^{1,2})) &\in \delta^4 \Omega^2(P)[\delta] \end{aligned}$$

The fact that both d_δ and d_δ^* of the polynomial vanish on the order $N + 2$ implies uniqueness of the terms of order up to δ^2 .

First attempt at giving power series

Note that $\alpha(\Theta) - \pi^* h \in \Omega^3(P)^G$ (Proposition 2.5.6), which will allow us to calculate in the isomorphic complexes (3.3.2) and (3.3.3). First, we ignore the $\psi^{1,2}$ and calculate $d_\delta(\alpha^{0,3} + \delta^2 \alpha^{2,1} - \delta^3 \pi^* h)$. As a reminder (Proposition 2.5.6),

$$d\alpha(\Theta) = \pi^* \langle \Omega \wedge \Omega \rangle \in \Omega^{4,0}(P).$$

Therefore,

$$d(\alpha(\Theta) - \pi^* h) = \pi^* (\langle \Omega \wedge \Omega \rangle - \langle \Omega \wedge \Omega \rangle) = 0.$$

From (4.2.1)

$$d_\delta = \rho_\delta d \rho_\delta^{-1} = d^{0,1} + \delta d^{1,0} + \delta^2 d^{2,-1},$$

and this implies that

$$d_\delta \rho_\delta (\alpha(\Theta) - \pi^* h) = d_\delta (\alpha^{0,3} + \delta^2 \alpha^{2,1} - \delta^3 \pi^* h) = 0. \quad (5.3.3)$$

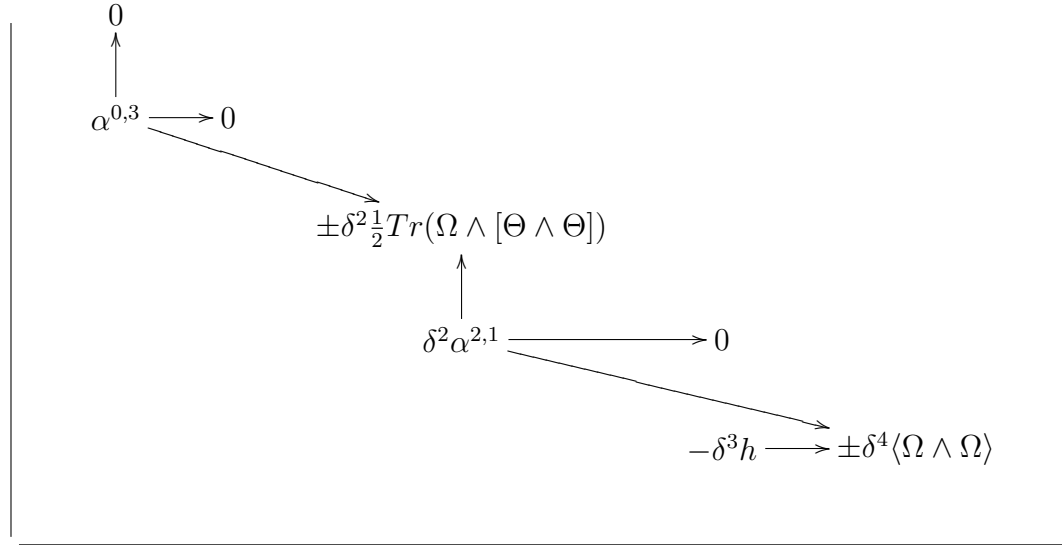
Thus, we also know how d_δ acts on the two components of $\alpha(\Theta)$. In particular, due to the bi-grading,

$$d^{0,1} \alpha^{0,3} = 0, \quad d^{1,0} \alpha^{0,3} = 0, \quad d^{1,0} \alpha^{2,1} = 0. \quad (5.3.4)$$

Under the isomorphism with (3.3.2), we can write this as

$$\begin{aligned} d_\delta (\alpha^{0,3} + \delta^2 \alpha^{2,1} - \delta^3 h) &= \delta^2 (\iota_\Omega \alpha^{0,3} - d_{\mathfrak{g}} \alpha^{2,1}) + \delta^4 (\iota_\Omega \alpha^{2,1} - d_M h) \\ &= \delta^4 (\langle \Omega \wedge \Omega \rangle - \langle \Omega \wedge \Omega \rangle) = 0 \end{aligned} \quad (5.3.5)$$

Visually, this can be pictured in (3.3.2), where each arrow is a (a priori) non-trivial component of d_δ , and the \pm sign is used if two elements cancel.



Now we wish to calculate $d_\delta^* (\alpha^{0,3} + \delta^2 \alpha^{2,1} - \delta^3 \pi^* h)$. Because we only need that terms of order δ^3 or less are zero, higher powers of δ are ignored. In the picture below, this is represented by making the corresponding arrows dotted. In particular, dotted arrows do not indicate the actual image of an element (as a form), only the bi-grading of where it lives.

When we restrict $\alpha^{0,3}$ to a fiber (Proposition 2.5.6), it is simply a multiple of the Ad -invariant form

$$\langle \theta \wedge [\theta \wedge \theta] \rangle \in \Lambda^3 \mathfrak{g}_P^*.$$

This implies that $\alpha^{0,3}$ is harmonic when restricted to fibers (Proposition 2.4.4). Therefore,

$$d_{\mathfrak{g}}^* \alpha^{0,3} = 0.$$

By assumption, $h \in d_M^* \Omega^4(M)$, and therefore $d_M^* h = 0$. Doing a visual calculation in the complex (3.3.3) gives ...

$$\begin{array}{c}
 \alpha^{0,3} \\
 \downarrow \\
 0 \longleftarrow \delta^4 \delta^2 \alpha^{2,1} \\
 \delta^3 \gamma^{1,1} \longleftarrow \delta^2 \alpha^{2,1} \\
 \delta^5 \delta^3 h \longleftarrow 0 \\
 \delta^3 h
 \end{array}$$

$$d_{\delta}^*(\alpha^{0,3} + \delta^2 \alpha^{2,1} - \delta^3 \pi^* h) = \delta^3 \gamma^{1,1} + O(\delta^4)$$

for some element $\gamma^{1,1} \in \Omega^1(M; \mathfrak{g}_P^*)$. Therefore, we must add in another element to cancel this term.

Note that since $d_{\mathfrak{g}}$ is linear over $\Omega^i(M)$ (3.1.2), then $d_{\mathfrak{g}}^*$ is linear over $\Omega^i(M)$ as well. Therefore, the homology of any vertical column

$$H_{\bullet}(\Omega^i(M; \Lambda^{\bullet} \mathfrak{g}_P^*), d_{\mathfrak{g}}^*) = \Omega^i(M) \otimes H_{\bullet}(\Lambda^{\bullet} \mathfrak{g}_P^*, d_{\mathfrak{g}}^*).$$

The Lie group G is simple, and therefore $H_1(\Lambda^{\bullet} \mathfrak{g}_P^*, d_{\mathfrak{g}}^*) \cong H^1(\Lambda^{\bullet} \mathfrak{g}_P^*, d_{\mathfrak{g}}^*) = 0$. Since

$d_{\mathfrak{g}}^* \gamma^{1,1} = 0$, this implies that $\gamma^{1,1} \in d_{\mathfrak{g}}^* \Omega^1(M; \Lambda^2 \mathfrak{g}_P^*)$. Define $\psi^{1,2} \in \Omega^1(M; \Lambda^2 \mathfrak{g}_P^*)$ to be the unique element, under the Hodge decomposition of $\Lambda^2 \mathfrak{g}_P^*$, such that

$$d_{\mathfrak{g}}^* \psi^{1,2} = \gamma^{1,1}; \quad d_{\mathfrak{g}} \psi^{1,2} = 0.$$

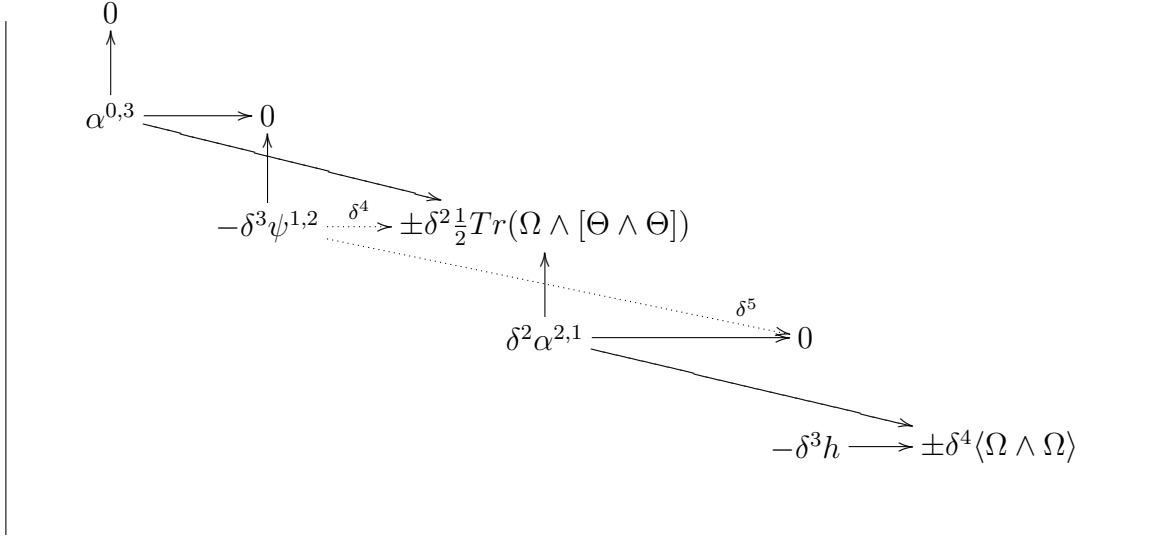
Successfully showing existence of power series

Now we show that

$$\begin{aligned} d_{\delta} (\alpha^{0,3} + \delta^2 \alpha^{2,1} - \delta^3 (\pi^* h + \psi^{1,2})) &\in \delta^4 \Omega^4(P)[\delta] \\ d_{\delta}^* (\alpha^{0,3} + \delta^2 \alpha^{2,1} - \delta^3 (\pi^* h + \psi^{1,2})) &\in \delta^4 \Omega^2(P)[\delta] \end{aligned}$$

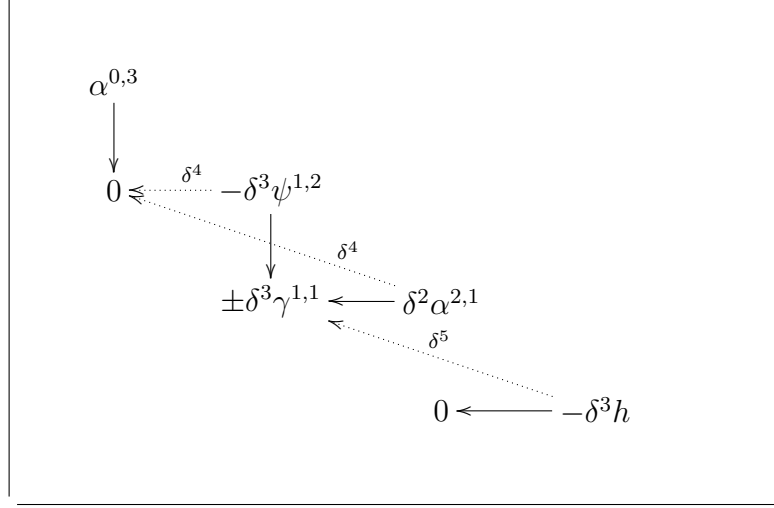
The d_{δ} calculation follows in the complex (3.3.2) as before, but with the added information that $d_{\mathfrak{g}} \psi^{2,1} = 0$:

$$\begin{aligned} d_{\delta} (\alpha^{0,3} + \delta^2 \alpha^{2,1} - \delta^3 (\pi^* h + \psi^{1,2})) &= \delta^2 (\iota_{\Omega} \alpha^{0,3} - d_{\mathfrak{g}} \alpha^{2,1}) - \delta^3 (d_{\mathfrak{g}} \psi^{1,2}) + O(\delta^4) \\ &= 0 + O(\delta^4). \end{aligned}$$



Likewise the d_δ^* follows as before, but with the $d_{\mathfrak{g}}^*\psi^{1,2}$ cancelling the $d_{\nabla}^*\alpha^{2,1}$ term:

$$\begin{aligned} d_\delta^*(\alpha^{0,3} + \delta^2\alpha^{2,1} - \delta^3(\pi^*h + \psi^{1,2})) &= d_{\mathfrak{g}}^*\alpha^{0,3} - \delta^3 d_M^*h + \delta^3(d_{\nabla}^*\alpha^{2,1} - d_{\mathfrak{g}}^*\psi^{1,2}) + O(\delta^4) \\ &= 0 + O(\delta^4). \end{aligned}$$



To show that a power series of the form $\alpha^{0,3} + \delta^2\alpha^{2,1} + O(\delta^3) \in E_{\infty,\delta}^{0,3}$, we must also show that $\alpha^{2,1} \in E_\infty^\perp$. In (5.3.1) we showed that

$$E_\infty^3 = E_2^{0,3} \oplus E_2^{3,0}.$$

Therefore, $\alpha^{2,1} \in \Omega^{2,1}(P) \subset E_\infty^\perp$. By Proposition 4.2.5, there exists a formal power series in $E_{\infty,\delta}^3$ whose terms of order $\delta \leq 2$ are equal to the above power series. We have then proved the existence of a power series of the form

$$\alpha^{0,3} + \delta^2\alpha^{2,1} + O(\delta^3) \in E_{\infty,\delta}^{0,3}.$$

This also implies that the constant term $\alpha^{0,3} \in E_\infty^{0,3}$. By (5.3.1),

$$\dim E_{\infty,\delta}^{0,3} = \dim H^3(G; \mathbb{R}) = 1,$$

and thus

$$E_\infty^{0,3} = \mathbb{R}[\alpha^{0,3}].$$

Properties of h term

We have proved there exist forms $\phi_3, \phi_4, \dots \in \Omega^3(P)$ such that

$$d_\delta(\alpha^{0,3} + \delta^2 \alpha^{2,1} + \delta^3 \phi_3 + \delta^4 \phi_4 + \dots) = 0 \quad (5.3.6)$$

$$d_\delta^*(\alpha^{0,3} + \delta^2 \alpha^{2,1} + \delta^3 \phi_3 + \delta^4 \phi_4 + \dots) = 0 \quad (5.3.7)$$

We now investigate the $(3,0)$ component of ϕ_3 , which we denote $\phi_3^{3,0}$. We first wish to see that $\phi_3^{3,0}$ is right-invariant. Analyzing the δ^3 component of (5.3.6), we see that

$$\delta^3(d^{1,0} \alpha^{2,1} + d^{0,1} \phi_3^{3,0}) = 0.$$

From earlier calculations in this proof (5.3.4), $d^{1,0} \alpha^{2,1} = 0$. Therefore,

$$d^{0,1} \phi_3^{3,0} = 0,$$

which implies that $\phi_3^{3,0}$ is constant along the fibers, and hence (Lemma 3.1.3)

$$\phi_3^{3,0} \in \Omega^3(P)^G.$$

Since $\phi_3^{3,0}$ is right-invariant, we can use the simpler complex (3.3.2) to see d_M and d_M^* . By analyzing the $(4,0)$ component of δ^4 in the first equation (5.3.6),

$$\delta^4(\iota_\Omega \alpha^{2,1} + d_M(\phi_3^{3,0})) = 0.$$

Therefore (by (5.3.5)), $d_M(\phi_3^{3,0}) = -\langle \Omega \wedge \Omega \rangle$. By analyzing the $(2,0)$ component of δ^4 in the second equation (5.3.7),

$$\delta^4 d_M^*(\phi_3^{3,0}) = 0.$$

Furthermore, we require that $\phi_3 \perp E_\infty$, so that we have the unique element in $E_{\infty,\delta}^3$ according to Theorem 4.2.6. Since $E_\infty^{3,0} = \pi^*\mathcal{H}_g^3(M)$, then

$$\phi_3^{3,0} \perp \mathcal{H}_g^3(M).$$

This uniquely characterizes the form $\phi_3^{3,0}$, which we denote $-\pi^*h$. Therefore, $h \in \Omega^3(M)$ with

$$dh = -\langle \Omega \wedge \Omega \rangle, \quad h \in d^*\Omega^4(M).$$

The power series is of the form

$$\alpha^{0,3} + \delta^2\alpha^{2,1} - \delta^3\pi^*h + O(\delta^3) \in E_{\infty,\delta}^{0,3}, \quad (5.3.8)$$

where the remainder of the δ^3 term is perpendicular to $\Omega^{3,0}(P)$. \square

Theorem 5.3.2. *Given (M, P, g, Θ) with G simple and $[\langle \Omega \wedge \Omega \rangle] = 0 \in H^4(M; \mathbb{R})$, then*

$$\text{Ker } \Delta_0^3 = \lim_{\delta \rightarrow 0} \text{Ker } \Delta_{g_\delta}^3 = \mathbb{R}[\alpha(\Theta) - \pi^*h] \oplus \pi^*\mathcal{H}_g^3(M) \subset \Omega^3(P).$$

Here, $\alpha(\Theta)$ is the Chern–Simons 3-form, and $h \in \Omega^3(M)$ is the unique form such that

$$dh = \langle \Omega \wedge \Omega \rangle, \quad h \in d^*\Omega^4(M).$$

Proof. By Proposition 5.3.1, $E_{\infty,\delta}^{0,3}$ is generated by an element of the form

$$\alpha^{0,3} + \delta^2\alpha^{2,1} - \delta^3\pi^*h + O(\delta^3),$$

where the remainder of the δ^3 term is orthogonal to $\Omega^{3,0}(P)$. This implies that ρ_δ^{-1} is defined on the above power series, since ρ_δ^{-1} divides by δ^i where i is the horizontal part of the bi-grading. By Proposition 4.3.3,

$$(\rho_\delta^{-1}(\alpha^{0,3} + \delta^2\alpha^{2,1} - \delta^3\pi^*h + O(\delta^3)))_{\delta=0} \in \text{Ker } \Delta_0^3,$$

and

$$\begin{aligned} (\rho_\delta^{-1}(\alpha^{0,3} + \delta^2\alpha^{2,1} - \delta^3\pi^*h + O(\delta^3)))_{\delta=0} &= (\alpha^{0,3} + \alpha^{2,1} - \pi^*h + O(\delta))_{\delta=0} \\ &= \alpha(\Theta) - \pi^*h. \end{aligned}$$

Proposition 5.3.1 also states that $E_\infty^{3,0} = \pi^*\mathcal{H}_g^3(M)$. Using Corollary 4.3.4, this gives us

$$\pi^*\mathcal{H}_g^3(M) \subset \text{Ker } \Delta_0^3.$$

We have now given an injective map $E_\infty^3 \hookrightarrow \text{Ker } \Delta_0^3$, and since these spaces are isomorphic, we have the equality

$$\text{Ker } \Delta_0^3 = \mathbb{R}[\alpha(\Theta) - \pi^*h] \oplus \pi^*\mathcal{H}_g^3(M).$$

□

Remark 5.3.3. *In the analogous situation for a $U(n)$ -bundle, we saw that the space of harmonic 1-forms on P is independent of scaling factor δ . In the above proof for harmonic 3-forms, one sees the possibility of higher order δ terms in $\text{Ker } \Delta_\delta$. In fact, these terms do appear in explicit calculations, and the description of harmonic forms in terms of the Chern–Simons form only holds in the adiabatic limit. It is not known what information the higher order δ terms contain.*

Theorem 5.3.4. *Given (M, P, g, Θ) with G simple and $[\langle \Omega \wedge \Omega \rangle] \neq 0 \in H^4(M; \mathbb{R})$, then*

$$\text{Ker } \Delta_0^3 = \lim_{\delta \rightarrow 0} \text{Ker } \Delta_{g_\delta}^3 = \pi^*\mathcal{H}_g^3(M) \subset \Omega^3(P).$$

Proof. This follows from (5.3.2) and the first part of Proposition 5.3.1:

$$E_\infty^3 = E_2^{3,0} = \pi^*\mathcal{H}_g^3(M).$$

Combining this with Corollary 4.3.4 gives us the inclusion

$$\pi^*\mathcal{H}_g^3(M) \subset \text{Ker } \Delta_0^3,$$

which is an equality since both vector spaces have the same dimension. \square

CHAPTER 6

CANONICAL FORMS ASSOCIATED TO LIFTS OF THE STRUCTURE GROUP

In this section, we define a string structure in an analogous way to a spin structure: it is a particular type of cohomology class on the total space P . A string structure is a class $\mathcal{S} \in H^3(P; \mathbb{Z})$ that restricts to the standard generator of $H^3(\text{Spin}(n); \mathbb{Z})$ on the fibers ($n \geq 5$). If we introduce a connection on P and a metric on M , we can take the harmonic representative of the (\mathbb{R} -image) of \mathcal{S} in the adiabatic limit. This harmonic representative is the Chern–Simons 3-form minus a 3-form $H \in \Omega^3(M)$. This form H has properties similar to the Chern–Simons 3-form. We should note that in this construction, we only consider the \mathbb{R} -image of a string structure and lose all information about the torsion in $H^3(P; \mathbb{Z})$. It is hoped that this construction can be improved so that the torsion information is not lost. Such a construction would most likely use some form of differential cohomology.

6.1 Generalities about G' structures

So far, we have dealt with principal G -bundles over a space M , where G is a compact Lie group. However, the notion of a principal bundle makes sense for more general topological groups. The general theory of fiber bundles tells us that for a topological group G , there is a universal G -bundle $EG \rightarrow BG$, where BG is called the classifying space of G . In other words, any G -bundle over M is the pullback

of a classifying map $M \rightarrow BG$. The spaces EG and BG are characterized by the following: given a contractible space EG on which G acts freely, let $BG \stackrel{\text{def}}{=} EG/G$. The bundle $G \hookrightarrow EG \rightarrow BG$ is then the universal G -bundle, and is unique up to homotopy. So, any G -bundle $P \rightarrow M$ is the pullback of a classifying map in the following diagram:

$$\begin{array}{ccc}
 G & \xlongequal{\quad} & G \\
 \downarrow & & \downarrow \\
 P & \xrightarrow{f^*} & EG \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{f} & BG
 \end{array}$$

This gives us a bijection

$$\{G \text{ bundles over } M\} \longleftrightarrow \text{Map}(M, BG).$$

In fact, any two bundles over M are isomorphic if and only if their classifying maps are homotopic ($f \simeq f'$). This gives a bijection

$$\{G \text{ bundles over } M\}/\text{iso} \longleftrightarrow [M, BG],$$

where the brackets denote homotopy classes of maps.

Given such a G -bundle, one often wants to make the bundle compatible with some additional structure. Common examples include a choice of orientation, spin structure, or complex structure. These can often be phrased in terms of reducing or “lifting” the structure group. Given a group homomorphism $G' \rightarrow G$, this induces a map $BG' \rightarrow BG$. Given a principal G -bundle P , then a “lift of the structure group to G' ” is a choice of lift \tilde{f}

$$\begin{array}{ccc}
 & & BG' \\
 & \nearrow \tilde{f} & \downarrow \\
 M & \xrightarrow{f} & BG
 \end{array}$$

The lift \tilde{f} induces a G' bundle which “covers” P in the following sense. Given

$$\begin{array}{ccccc}
 & & & & G' \\
 & & & & \downarrow \\
 & & & & EG' \\
 & & & \swarrow & \downarrow \\
 G & \xlongequal{\quad} & G & & EG' \\
 \downarrow & & \downarrow & \swarrow & \downarrow \\
 P & \xrightarrow{f^*} & EG & \xrightarrow{\tilde{f}} & BG' \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 M & \xrightarrow{f} & BG & &
 \end{array}$$

we obtain

$$\begin{array}{ccc}
 G' & \longrightarrow & G \\
 \downarrow & & \downarrow \\
 P' & \longrightarrow & P \\
 & \searrow & \downarrow \\
 & & M
 \end{array}$$

The most basic examples are given by inclusions $G' \hookrightarrow G$. In this case, such a lift is often called a reduction of the structure group to G' . For example, An orientation is a reduction of the structure group to $SO(n) \hookrightarrow O(n)$. Another fundamental but more subtle example is a spin structure. On a principal $SO(n)$ -bundle $P \rightarrow M$, a choice of a spin structure is a lift

$$\begin{array}{ccc}
 & & BSpin(n) \\
 & \nearrow \tilde{f} & \downarrow \\
 M & \xrightarrow{f} & BSO(n)
 \end{array}$$

Here, the map $BSpin(n) \rightarrow BSO(n)$ is induced from the double cover $Spin(n) \rightarrow SO(n)$.

For simplicity, we restrict to $n \geq 3$, which is the stable range for $\pi_1(SO(n))$.

Then, the homomorphism $Spin(n) \rightarrow SO(n)$ is special in that

$$\pi_i(Spin(n)) = \begin{cases} \pi_i(SO(n)) & i > 1 \\ 0 & i = 0, 1 \end{cases}$$

In other words, $Spin(n)$ is the 1-connected cover of $SO(n)$. Furthermore, in the language of bundles, we know that $Spin(n)$ is a $\pi_1(SO(n)) \cong \mathbb{Z}/2$ bundle over $SO(n)$. Therefore, up to homotopy, $Spin(n)$ is simply the pullback of a map $SO(n) \rightarrow B\mathbb{Z}/2$.

$$\begin{array}{ccc} \mathbb{Z}/2 & \xlongequal{\quad} & \mathbb{Z}/2 \\ \downarrow & & \downarrow \\ Spin(n) & \xrightarrow{f^*} & E\mathbb{Z}/2 \\ \downarrow & & \downarrow \\ SO(n) & \xrightarrow{f} & B\mathbb{Z}/2 \end{array}$$

Since $\mathbb{Z}/2$ is a discrete group, as a space it is the Eilenberg-MacLane space $K(\mathbb{Z}/2, 0)$.

Eilenberg-MacLane spaces are characterized by the property

$$\pi_i(K(H, n)) = \begin{cases} H & i = n \\ 0 & i \neq n \end{cases}$$

for any abelian group H and $n \in \mathbb{N}$. They are also the representing spaces for the ordinary cohomology spectrum, giving

$$H^n(X; H) = [X, K(H, n)].$$

They are also topological groups and are compatible with the B functor in the sense that

$$BK(H, n) \simeq K(H, n + 1).$$

Viewed in this language, we see that up to homotopy, $Spin(n)$ is the pullback of a

map $SO(n) \xrightarrow{f} BK(\mathbb{Z}/2, 0) \simeq K(\mathbb{Z}/2, 1)$

$$\begin{array}{ccc}
 K(\mathbb{Z}/2, 0) & \xlongequal{\quad} & K(\mathbb{Z}/2, 0) \\
 \downarrow & & \downarrow \\
 Spin(n) & \xrightarrow{f^*} & EK(\mathbb{Z}/2, 0) \\
 \downarrow & & \downarrow \\
 SO(n) & \xrightarrow{f} & K(\mathbb{Z}/2, 1)
 \end{array}$$

The map $SO(n) \xrightarrow{f} K(\mathbb{Z}/2, 1)$ gives the generator of $H^1(SO(n); \mathbb{Z}/2)$.

Applying B to the above bundle gives a description of $BSpin(n)$ as the pullback of Bf in the diagram

$$\begin{array}{ccc}
 K(\mathbb{Z}/2, 1) & \xlongequal{\quad} & K(\mathbb{Z}/2, 1) \\
 \downarrow & & \downarrow \\
 BSpin(n) & \xrightarrow{Bf^*} & EK(\mathbb{Z}/2, 1) \\
 \downarrow & & \downarrow \\
 BSO(n) & \xrightarrow{Bf} & K(\mathbb{Z}/2, 2)
 \end{array}$$

The map Bf represents the universal second Stieffel-Whitney class $w_2 \in H^2(BSO(n); \mathbb{Z}/2)$.

One may feel this abstraction is unnecessary, as we have explicit descriptions of $Spin(n)$ without all this topology language. However, it is not clear how to best think about the 3-connected cover of $Spin(n)$, which we denote $String(n)$ (when $n \geq 5$). Yet, there is interesting cohomological information that is seen by only considering this topological group up to homotopy.

We now describe a standard generalization of the above construction and think about lifts of the structure group to a topological group G' , where G' is, up to homotopy, the homotopy fiber of a map $G \rightarrow K(H, n)$. Note that we always assume the existence of the topological groups G and G' , and then consider their homotopy type.

Let K be a path-connected CW complex, and fix a basepoint $k_0 \in K$. Then,

define PK , the **path-space** of K , along with a map $\pi : PK \rightarrow K$ by

$$PK \stackrel{\text{def}}{=} \{\gamma : [0, 1] \rightarrow K \mid \gamma(0) = k_0\} \xrightarrow{\pi} K$$

$$\gamma \longmapsto \gamma(1)$$

While the pathspace PK is contractible, the fiber of $\pi : PK \rightarrow K$ at a point $k_1 \in K$ is

$$PK_{k_1} = \pi^{-1}(k_1) = \{\gamma : [0, 1] \rightarrow K \mid \gamma(0) = k_0, \gamma(1) = k_1\},$$

which is homotopy equivalent to the pointed loop space

$$\Omega K \stackrel{\text{def}}{=} \{\gamma : [0, 1] \rightarrow K \mid \gamma(0) = \gamma(1) = k_0\}.$$

Definition 6.1.1. Given a map $f : Y \rightarrow K$, we define the **homotopy fiber** Y' to be the pullback of f in the diagram

$$\begin{array}{ccc} \Omega K & \xlongequal{\quad} & \Omega K \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{f^*} & PK \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & K \end{array}$$

Proposition 6.1.2. Suppose Y' is the homotopy fiber of $Y \rightarrow K(H, n)$. Then,

$$\pi_i(Y') \cong \pi_i(Y); \quad i \neq n, n - 1$$

Furthermore, if $Y \rightarrow K(H, n)$ is the Hurewicz image of $\pi_n(Y)$, then

$$\pi_i(Y') = \begin{cases} \pi_i(Y) & i \neq n \\ 0 & i = n \end{cases}$$

Proof. The proof follows directly from the long exact sequence of homotopy groups associated to the fibration. □

Proposition 6.1.3. *There is a natural isomorphism $\pi_{k+1}BG \rightarrow \pi_k G$ induced from the fibration $G \rightarrow EG \rightarrow BG$.*

We now consider the situation where G' is, up to homotopy, the homotopy fiber of a map $G \rightarrow K(H, n)$, where $K(H, n)$ is an Eilenberg-MacLane space and H some abelian group. Let BG' be the corresponding classifying space, which is the homotopy fiber of the induced map $BG \rightarrow K(H, n + 1)$. In other words, BG' is obtained from “killing” a cohomology class. Let $\mathcal{C} \in H^n(G; H)$ and $B[\mathcal{C}] \in H^{n+1}(BG; H)$ denote the cohomology classes associated to the maps $\mathcal{C} : G \rightarrow K(H, n)$ and $B\mathcal{C} : BG \rightarrow K(H, n + 1)$.

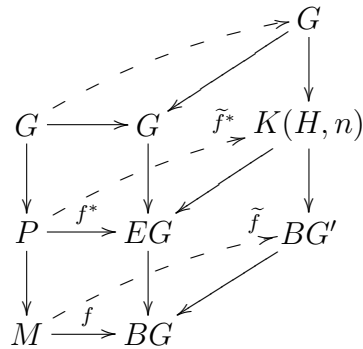
Proposition 6.1.4. *Let $P \rightarrow M$ be a principal G -bundle, classified by a map $f : M \rightarrow BG$. The following are equivalent:*

- *a homotopy class of lifts $\tilde{f} : M \rightarrow BG'$ of the fixed classifying map f .*
- *a cohomology class $\mathcal{G}' \in H^n(P; H)$ such that $i^*\mathcal{G}' = \mathcal{C} \in H^n(G; H)$.*

Proof. The proof follows from the fact that iterating the homotopy fiber construction gives us the sequence

$$\cdots \rightarrow G' \rightarrow G \rightarrow K(H, n) \rightarrow BG' \rightarrow BG \rightarrow K(H, n + 1)$$

where any two consecutive maps are a fibration. The homotopy fiber of $BG' \rightarrow BG$ is $K(H, n)$. The existence of \tilde{f} allows us to perform a lift of maps between fibrations



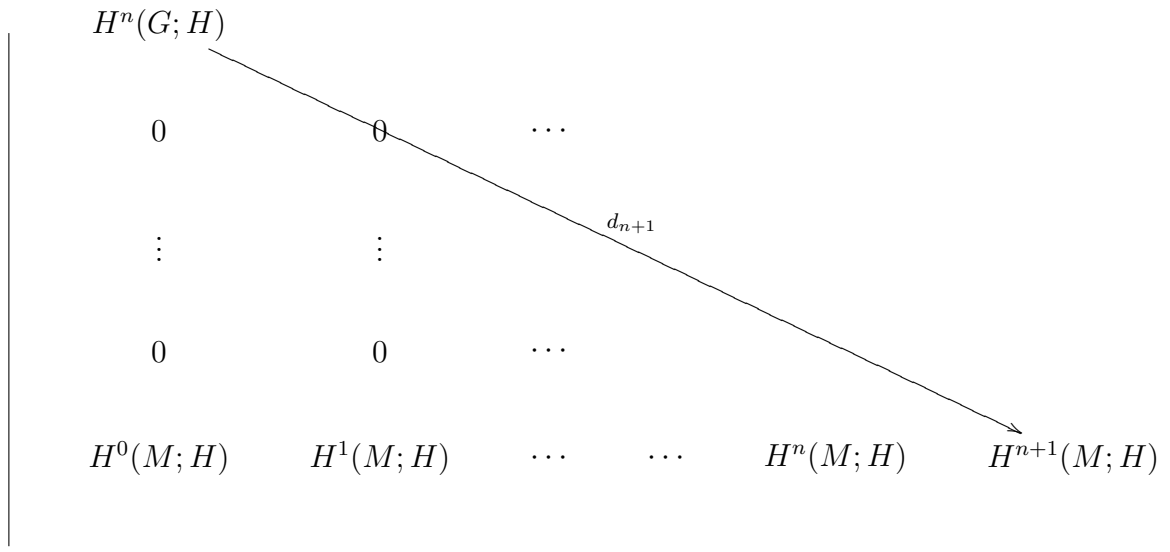
Therefore, we see a map $\tilde{f}^* : P \rightarrow K(H, n)$ that restricts to our preferred map $G \rightarrow K(H, n)$ on the fibers. This gives an element $\mathcal{G}' \in H^n(P; H)$ that restricts to the canonical element \mathcal{C} on the fibers. Any two lifts \tilde{f} which are homotopic induce the same cohomology class.

Conversely, a cohomology class $\mathcal{G}' \in H^n(P; H)$ restricting to \mathcal{C} on the fibers gives, up to homotopy, the map $\tilde{f}^* : P \rightarrow K(H, n)$ restricting to $G \rightarrow K(H, n)$ on the fibers. Such a map \tilde{f}^* induces \tilde{f} . \square

Proposition 6.1.5. *Assume $\tilde{H}^i(G; H) = 0$ for $i < n$. Given a principal bundle $G \rightarrow P \rightarrow M$ as in Proposition 6.1.4. Then*

1. *a G' structure exists if and only if $B\mathcal{C} \circ f \simeq *$, or equivalently, the characteristic class of P corresponding to $B\mathcal{C}$ is 0.*
2. *the space of G' structures is a torsor for $H^n(M; H)$.*

Proof. The proof follows directly from the Leray–Serre spectral sequence for the fibration. The $E_2 = \dots = E_{n+1}$ pages are equal, and the only non-trivial differential is the following d_{n+1}



This induces the exact sequence

$$0 \longrightarrow H^n(M; H) \xrightarrow{\pi^*} H^n(P; H) \xrightarrow{i^*} H^n(G; H) \xrightarrow{d_{n+1}} H^{n+1}(M; H)$$

$$\mathcal{G}' \overset{?}{\dashrightarrow} \mathcal{C} \longrightarrow BC(P)$$

Therefore, there exists such a G' structure \mathcal{G}' if and only if \mathcal{C} maps to 0. Furthermore, any two such \mathcal{G}' structures will differ by an element of $H^n(M; H)$. \square

Proposition 6.1.6. *Assume G is $(n - 1)$ connected (i.e. that $\pi_i(G) = 0$ for $i < n$) and that \mathcal{C} is the Hurewicz image of $\pi_n G$. Then, we have the additional equivalence for $P \rightarrow M$ classified by $f : M \rightarrow BG$:*

- a homotopy class of lifts $\tilde{f} : M \rightarrow BG'$ of the classifying map f .
- a trivialization (global frame) of P on the n skeleton of M that extends to the $n + 1$ skeleton. Trivializations are considered up to homotopy.

Proof. The hypotheses imply that G' is the n -connected cover of G , and BG' is the $(n + 1)$ -connected cover of BG . Therefore, given a lift $\tilde{f} : X \rightarrow BG'$, \tilde{f} is nullhomotopic on the $(n + 1)$ -skeleton of X , and any two nullhomotopies on the n -skeleton are homotopic to each other. Projecting these homotopies to $X \rightarrow BG$, we see that the first condition implies the second. Conversely, if classifying map $f : X \rightarrow BG$ is nullhomotopic on the $(n + 1)$ -skeleton, then the map $\mathcal{C} \circ f$ is nullhomotopic on all of X . A particular choice of nullhomotopy of $\mathcal{C} \circ f$ gives a lift to $X \rightarrow PK(H, n + 1)$, which in turn pulls back to a lift $\tilde{f} : X \rightarrow BG'$.

$$\begin{array}{ccccc}
 & & K(H, n) & \xlongequal{\quad} & K(H, n) \\
 & & \downarrow & & \downarrow \\
 & & BG' & \xrightarrow{\mathcal{C}^*} & PK(H, n + 1) \\
 \tilde{f} \nearrow & & \downarrow & \nearrow & \downarrow \\
 X & \xrightarrow{f} & BG & \xrightarrow{\mathcal{C}} & K(H, n + 1)
 \end{array}$$

□

6.2 \tilde{U} structures on $U(n)$ -bundles

Suppose we have a $U(n)$ -principal bundle P over a manifold M . For $n \geq 1$,

$$\pi_1(U(n)) \cong H^1(U(n); \mathbb{Z}) \cong \mathbb{Z},$$

and we consider the universal cover $\tilde{U}(n) \rightarrow U(n)$. Up to homotopy, this is just the fiber of the standard map $U(n) \rightarrow K(\mathbb{Z}, 1)$ that generates the first cohomology:

$$\tilde{U}(n) \rightarrow U(n) \rightarrow K(\mathbb{Z}, 1).$$

Remark 6.2.1. *The inclusion $U(1) \hookrightarrow U(n)$ gives an isomorphism on the first cohomology, and the standard orientation of $U(1)$ picks out a canonical generator of $H^1(U(1); \mathbb{Z})$. This gives the standard generator for $H^1(U(n); \mathbb{Z})$.*

Remark 6.2.2. *The group $\tilde{U}(n)$ has an explicit description as the semi-direct product $SU(n) \ltimes \mathbb{R}$.*

Definition 6.2.3. ($n \geq 1$) *A \tilde{U} structure on a $U(n)$ -bundle $P \rightarrow M$ is a cohomology class $\tilde{\mathcal{U}} \in H^1(P; \mathbb{Z})$ such that $i^*\tilde{\mathcal{U}} \in H^1(U(n); \mathbb{Z})$ is the standard generator, where $i : U(n) \hookrightarrow P$ is the fiber-wise inclusion.*

Proposition 6.2.4. *A \tilde{U} structure is equivalent to*

- a homotopy class of lifts \tilde{f}

$$\begin{array}{ccc} & & B\tilde{U}(n) \\ & \nearrow \tilde{f} & \downarrow \\ M & \xrightarrow{f} & BU(n) \end{array}$$

- a homotopy class of trivializations of P on the 1-skeleton that extend to the 2-skeleton of M .

A \tilde{U} structure exists if and only if $c_1(P) = 0 \in H^2(M; \mathbb{Z})$. Furthermore, the space of \tilde{U} structures is a torsor for $H^1(M; \mathbb{Z})$.

Proof. These statements follow from Propositions 6.1.4, 6.1.5, and 6.1.6. \square

Theorem 6.2.5. *Let P be a principal $U(n)$ -bundle with connection Θ over a Riemannian manifold (M, g) . If $\tilde{\mathcal{U}} \in H^1(P; \mathbb{Z})$ is a choice of \tilde{U} -structure, then for any $\delta \geq 0$,*

$$[\tilde{\mathcal{U}}]_{g_\delta} = \frac{i}{2\pi} \text{Tr}(\Theta) - \pi^* H_{g, \Theta, \tilde{\mathcal{U}}} \in \Omega^1(P).$$

The form $H_{g, \Theta, \tilde{\mathcal{U}}} \in \Omega^1(M)$ satisfies $dH_{g, \Theta, \tilde{\mathcal{U}}} = c_1(P, \Theta)$ and $d^ H_{g, \Theta, \tilde{\mathcal{U}}} = 0$. Furthermore, if the \tilde{U} structure is changed by $\xi \in H^1(M; \mathbb{Z})$, then*

$$[\tilde{\mathcal{U}} + \pi^* \xi]_{g_\delta} = [\tilde{\mathcal{U}}]_{g_\delta} + \pi^* [\xi]_g.$$

Proof. The existence of a \tilde{U} structure implies that $c_1(P) = 0 \in H^2(M; \mathbb{R})$. Therefore, Theorem 5.1.3 says that

$$\text{Ker } \Delta_{g_\delta}^1 = \mathbb{R} \left[\frac{i}{2\pi} \text{Tr}(\Theta) - \pi^* h \right] \oplus \pi^* \mathcal{H}_g^1(M),$$

where $dh = c_1(P, \Theta)$ and $h \in d^* \Omega^2(M)$. The cohomology class $\tilde{\mathcal{U}}$ restricts to the generator in $H^1(U(n); \mathbb{Z})$. Therefore, the harmonic representative $[\tilde{\mathcal{U}}]_{g_\delta}$ must be of the form

$$[\tilde{\mathcal{U}}]_{g_\delta} = 1 \left[\frac{i}{2\pi} \text{Tr}(\Theta) - \pi^* h \right] + \pi^*(h')$$

where $h' \in \mathcal{H}_g^1(M)$. Setting $H_{g, \Theta, \tilde{\mathcal{U}}} = h - h'$, we see that

$$[\tilde{\mathcal{U}}]_{g_\delta} = \frac{i}{2\pi} \text{Tr}(\Theta) - \pi^* h + \pi^* h' = \frac{i}{2\pi} \text{Tr}(\Theta) - \pi^* H_{g, \Theta, \tilde{\mathcal{U}}}.$$

Finally, any two \tilde{U} structures $\tilde{\mathcal{U}}_1, \tilde{\mathcal{U}}_2$ are related by

$$\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 = \pi^* \xi$$

for some $\xi \in H^1(M; \mathbb{Z})$. By the description of harmonic forms on P , we see that the harmonic representative of $\pi^*\xi \in H^1(P; \mathbb{R})$ is the pullback of $[\xi]_g \in \mathcal{H}_g^1(M)$. \square

6.3 Spin^c structures on $SO(n)$ -bundles

For the following, assume $n \geq 3$. Note that

$$H^2(SO(n); \mathbb{Z}) \cong H_1(SO(n); \mathbb{Z}) \cong \mathbb{Z}/2.$$

The group $\text{Spin}^c(n)$ is, up to homotopy, the homotopy fiber of the map $SO(n) \rightarrow K(\mathbb{Z}, 2)$ generated by the non-trivial cohomology class. Note that spin^c is not simply-connected, but instead is an S^1 extension of $SO(n)$ (since $K(\mathbb{Z}, 2) \simeq BU(1)$).

Definition 6.3.1. ($n \geq 3$). A spin^c structure on a $SO(n)$ -bundle $P \rightarrow M$ is a cohomology class $c \in H^2(P; \mathbb{Z})$ such that $i^*c \neq 0 \in H^2(SO(n); \mathbb{Z}) \cong \mathbb{Z}/2$, where $i : SO(n) \hookrightarrow P$ is the fiber-wise inclusion.

Proposition 6.3.2. A spin^c structure on $P \rightarrow M$ is equivalent to a homotopy class of lifts \tilde{f}

$$\begin{array}{ccc} & & B\text{Spin}^c(n) \\ & \nearrow \tilde{f} & \downarrow \\ M & \xrightarrow{f} & SO(n) \end{array}$$

A spin^c structure exists if and only if $W_3(P) = 0 \in H^3(M; \mathbb{Z})$, where W_3 is the third integral Stiefel-Whitney class (obtained by the Bockstein of w_2). The space of spin^c structures is a torsor for $H^2(M; \mathbb{Z})$.

Proof. These statements follow from Propositions 6.1.4 and 6.1.5, along with the fact that in the universal case, the non-trivial element in $H^2(SO(n); \mathbb{Z})$ transgresses to $W_3 \in H^3(BSO(n); \mathbb{Z})$ via the Leray–Serre spectral sequence. \square

Theorem 6.3.3. ($n \geq 3$) *Let P be a principal $SO(n)$ -bundle with connection Θ over a Riemannian manifold (M, g) . If $c \in H^2(P; \mathbb{Z})$ is a choice of spin^c structure, then*

$$[c]_0 = \lim_{\delta \rightarrow 0} [c]_{g_\delta} = \pi^* H_{g, \Theta, c} \in \Omega^2(P).$$

Here, $H_{g, \Theta, c} \in \mathcal{H}_g^2(M)$. Furthermore, if the spin^c structure is changed by $\xi \in H^2(M; \mathbb{Z})$, then

$$[c + \pi^* \xi]_0 = [c]_0 + \pi^* [\xi]_g.$$

Proof. Since $H^2(SO(n); \mathbb{R}) = 0$, Theorem 5.2.5 says that

$$\text{Ker } \Delta_0^2 = \pi^* \mathcal{H}_g^2(M)$$

and hence $[c]_0 \in \pi^* \mathcal{H}_g^2(M)$. Furthermore, for $\xi \in H^2(M; \mathbb{Z})$, the adiabatic harmonic representative of the class pulled back to P is merely the pullback of the harmonic representative on M :

$$[\pi^* \xi]_0 = \pi^* [\xi]_g \in \Omega^2(P).$$

□

The above Theorem is interesting in the following way. A spin^c structure, topologically, is a choice of a particular complex line bundle $L \rightarrow P$, characterized by its first Chern class $c \in H^2(P; \mathbb{Z})$. To any spin^c structure, there is an associated line bundle λ on M . This is determined by requiring that

$$\pi^* c_1(\lambda) = 2c \in H^2(P; \mathbb{Z}).$$

In usual geometric applications, one must also choose a connection on the associated line bundle λ . However, if we introduce a metric g on M (along with a connection on P , which is usually already determined by g), then taking the harmonic representative of c in the adiabatic limit produces a canonical 2-form $H_{g, c} \in \Omega^2(M)$.

Furthermore, we see that $\pi^*c_1(\lambda) = 2[H] \in H^2(P; \mathbb{R})$, and hence $2H_{g,c} \in \Omega^2(M)$ is the curvature for some connection on λ . Furthermore, $H_{g,c}$ is harmonic, and so picks out a class of connections on λ with energy-minimizing curvature. In applications, it is often preferable to use such connections.

6.4 String structures on $Spin(n)$ -bundles

For now, assume $n \geq 5$. We will treat the case $n \leq 4$ in a moment. For $n \geq 5$, $Spin(n)$ is 2-connected and

$$\pi_3(Spin(n)) \cong H^3(Spin(n); \mathbb{Z}) \cong \mathbb{Z}.$$

Definition 6.4.1. *String(n) is a topological group that is the 3-connected cover of Spin(n). Up to homotopy, it is the homotopy fiber of the canonical map $Spin(n) \rightarrow K(\mathbb{Z}, 3)$.*

The group $String(n)$ does exist, and one construction can be found in section 5.4 of [ST]. It should be noted that $String(n)$ cannot be a compact Lie group, so any known model forces us to deal with more sophisticated ideas. Here, we can avoid this issue by simply assuming its existence and then thinking about it only up to homotopy. In particular, whatever model of $String(n)$ is used, a string structure will still give you the following cohomology class.

Definition 6.4.2. ($n \geq 5$) *A string structure on a $Spin(n)$ -bundle $P \rightarrow M$ is a cohomology class $\mathcal{S} \in H^3(P; \mathbb{Z})$ such that $i^*\mathcal{S} \in H^3(Spin(n); \mathbb{Z})$ is the standard generator, where $i : Spin(n) \hookrightarrow P$ is the fiber-wise inclusion.*

Proposition 6.4.3. *A string structure on $P \rightarrow M$ is equivalent to*

1. a homotopy class of lifts \tilde{f}

$$\begin{array}{ccc}
 & & BString(n) \\
 & \nearrow \tilde{f} & \downarrow \\
 M & \xrightarrow{f} & BSpin(n)
 \end{array}$$

2. a homotopy class of trivializations of P on the 3-skeleton of M which extend to the 4-skeleton.

Furthermore, a string structure exists if and only if $\frac{p_1}{2}(P) = 0 \in H^4(M; \mathbb{Z})$, and the space of string structures is a torsor for $H^3(M; \mathbb{Z})$.

Proof. These statements follow from Propositions 6.1.4, 6.1.5, and 6.1.6, along with the fact that in the universal case, the generator of $H^3(Spin(n); \mathbb{Z})$ transgresses to $\frac{p_1}{2} \in H^4(BSpin(n); \mathbb{Z})$ in the Leray–Serre spectral sequence. \square

The notion of a string structure is based on the obstruction $\frac{p_1}{2}(P) \in H^4(M; \mathbb{Z})$. As discussed at the end of Chapter 2, the $\frac{p_1}{2}$ class is a stable characteristic class. The definition of $String(n)$ holds for all n , but it is only for $n \geq 5$ that we have the nice description of $String(n)$ as being the 3-connected cover of $Spin(n)$. This is due to the fact that $\pi_3(Spin(n)) \cong H^3(Spin(n); \mathbb{Z})$ stabilizes at $n = 5$. This same thing happens when defining the groups $Spin(n)$ and spin structures. The groups $Spin(n)$ are always double-covers of $SO(n)$, but only for $n \geq 3$ does this have the description of being the universal cover.

Therefore, we say that a string structure for $n \leq 4$ is a cohomology class $\mathcal{S} \in H^3(P; \mathbb{Z})$ such that $i_x^* \mathcal{S} \in H^3(Spin(n); \mathbb{Z})$ is the image of the standard generator of $H^3(Spin(5); \mathbb{Z})$ under the inclusion $Spin(n) \hookrightarrow Spin(5)$. Alternatively, given a $Spin(n)$ -bundle with $n \leq 4$, we can look at the induced $Spin(5)$ -bundle and talk about string structures on it. Since this is also how the groups $String(n)$ are characterized for $n \leq 4$, a string structure is still equivalent to a homotopy class of lifts of the classifying map to $BString(n)$, and it is also equivalent to a homotopy class of trivializations of the stable bundle of P on the 3-skeleton which extends to the 4-skeleton.

Theorem 6.4.4. *Let P be a principal $Spin(n)$ -bundle with connection Θ over a Riemannian manifold (M, g) . If $\mathcal{S} \in H^3(P; \mathbb{Z})$ is a choice of string structure, then the adiabatic limit of the harmonic representative is given by*

$$[\mathcal{S}]_0 = \lim_{\delta \rightarrow 0} [\mathcal{S}]_{g_\delta} = \alpha(\Theta) - \pi^* H_{g, \Theta, \mathcal{S}} \in \Omega^3(P).$$

The form $H_{g, \Theta, \mathcal{S}} \in \Omega^3(M)$ has the properties $dH_{g, \Theta, \mathcal{S}} = \frac{p_1}{2}(P, \Theta)$, $d^ H_{g, \Theta, \mathcal{S}} = 0$. Furthermore, if the string structure is changed by $\xi \in H^3(M; \mathbb{Z})$, then*

$$[\mathcal{S} + \pi^* \xi]_0 = [\mathcal{S}]_0 + \pi^* [\xi]_g.$$

Proof. The existence of a string structure implies that $\frac{p_1}{2}(P) = 0 \in H^4(M; \mathbb{R})$. Therefore, Theorem 5.3.2 says that

$$\text{Ker } \Delta_0^3 = \mathbb{R}[\alpha(\Theta) - \pi^* h] \oplus \pi^* \mathcal{H}_g^3(M),$$

where $dh = \frac{p_1}{2}(P, \Theta)$ and $h \in d^* \Omega^4(M)$. The cohomology class \mathcal{S} restricts to the generator in $H^3(Spin(n); \mathbb{Z})$. Therefore, the harmonic representative $[\mathcal{S}]_0$ must be of the form

$$[\mathcal{S}]_0 = 1[\alpha(\Theta) - \pi^* h] + \pi^*(h')$$

where $h' \in \mathcal{H}_g^3(M)$. Setting $H = h - h'$, we see that

$$[\mathcal{S}]_0 = \alpha(\Theta) - \pi^* h + \pi^* h' = \alpha(\Theta) - \pi^* H_{g, \Theta, \mathcal{S}}.$$

Finally, for $\xi \in H^4(M; \mathbb{Z})$, the adiabatic harmonic representative of the class pulled back to P is merely the pullback of the harmonic representative on M :

$$[\pi^* \xi]_0 = \pi^* [\xi]_g \in \Omega^4(P).$$

□

It may be tempting to think, at least rationally, that there is a *canonical* string

structure determined by the real cohomology class $[\alpha(\Theta) - \pi^*h]$. However, this is not necessarily an *integral* class. In general, some closed form from the base must be added so that the resulting form has integral periods. Consequently, the canonical form $H_{g,\Theta,s} \in \Omega^3(M)$ associated to a string structure normally does not have integral periods. In fact, this has a natural interpretation in terms of classical Chern–Simons theory.

A global section $p : M \xrightarrow{\pi} P$ gives an isomorphism of principal bundles

$$P \underset{p}{\cong} M \times Spin(n).$$

Via the Künneth formula, along with the fact that $H^1(Spin(n); \mathbb{Z}) = H^2(Spin(n); \mathbb{Z}) = 0$, the above isomorphism induces the following isomorphism on integral cohomology,

$$H^3(P; \mathbb{Z}) \cong H^3(M; \mathbb{Z}) \oplus H^3(Spin(n); \mathbb{Z})$$

$$\mathfrak{S} \leftrightarrow (0, 1_{Spin})$$

and we say the string structure determined by p is the element $\mathfrak{S} \in H^3(P; \mathbb{Z})$ which corresponds to the pullback of the generator $1_{Spin} \in H^3(G; \mathbb{Z})$. In the following proposition, we let $H_s = H_{g,\Theta,s}$.

Proposition 6.4.5. *Let H_s be given by $(M, P, g, \Theta, \mathfrak{S})$. If $p : M \rightarrow P$ is a global section with induced string structure $\mathfrak{S} \in H^3(P; \mathbb{Z})$, then*

$$[p^*\alpha(\Theta) - H_s] = 0 \in H^3(M; \mathbb{R}).$$

Consequently, if $d^(p^*\alpha(\Theta)) = 0$, then $p^*\alpha(\Theta) = H_\xi$.*

Proof. First,

$$d(p^*\alpha(\Theta) - H_s) = \frac{p_1}{2}(\Theta) - \frac{p_1}{2}(\Theta) = 0.$$

Therefore, $[p^*\alpha(\Theta) - H_S] \in H^3(M; \mathbb{R})$. We now show that it evaluates as 0 on any 3-cycle X in M . Let $X \in \text{Map}(X^3, M)$ where X^3 is a 3-dimensional manifold. We use the notation \int_X to denote evaluation on the 3-cycle X . Then, under the isomorphism $P \cong M \times G$ induced by the global section p ,

$$p(X) \subset M \times \{pt\} \subset P.$$

Since $\xi \in H^3(P; \mathbb{Z})$ is zero on 3-cycles in $M \subset P$,

$$\langle \xi, p(X) \rangle = 0.$$

This induces the relationship in real cohomology

$$0 = \int_{p(X)} [\xi]_0 = \int_{p(X)} (\alpha(\Theta) - \pi^* H_S) = \int_X (p^*\alpha(\Theta) - H_S).$$

This holds for an arbitrary 3-cycle, and hence $[p^*\alpha(\Theta) - H_S] = 0 \in H^3(M; \mathbb{R})$.

Furthermore,

$$d^* H_S = 0$$

by Theorem 6.4.4. Therefore, if $d^*(p^*\alpha(\Theta)) = 0$, then $p^*\alpha(\Theta) - H_S$ is harmonic and consequently equals 0. In other words, under the Hodge decomposition, the forms $p^*\alpha(\Theta)$ and H_S are equal on the coexact and harmonic components. If $d^*(\alpha(\Theta)) = 0$, then the forms are equal. \square

6.5 Canonical forms on base from G' structures

Let $P \rightarrow M$ be a principal G -bundle, where G is appropriate to our situation. If $\text{Met}(M)$ is the space of Riemannian metrics on M , and \mathcal{A}_P is the affine space of connections on P , then we have canonical maps given by looking at the term from

the base in the previous adiabatic limit constructions.

$$\begin{aligned} \{\tilde{U} \text{ Structures}\} \times \text{Met}(M) \times \mathcal{A}_P &\longrightarrow \Omega^1(M) \\ \{\text{Spin}^c \text{ Structures}\} \times \text{Met}(M) \times \mathcal{A}_P &\longrightarrow \Omega^2(M) \\ \{\text{String Structures}\} \times \text{Met}(M) \times \mathcal{A}_P &\longrightarrow \Omega^3(M) \end{aligned}$$

Also, these maps are equivariant with respect to the action of $H^1(M; \mathbb{Z})$, $H^2(M; \mathbb{Z})$, and $H^3(M; \mathbb{Z})$ respectively, where $H^i(M; \mathbb{Z})$ acts freely and transitively on the space of Structures by π^* , and acts on $\Omega^i(M)$ by the addition of harmonic representatives.

Often the bundle P is determined by the metric g . In particular, frame bundles such as $\text{Spin}(M)$ depend on the metric and have a canonical connection determined by the metric g . In the case of a manifold M with spin structure, we obtain a map

$$\{\text{String Structures}\} \times \text{Met}(M) \longrightarrow \Omega^3(M)$$

by applying the construction to $(M, \text{Spin}(M), g, \Theta_{LC})$, where Θ_{LC} is the Levi-Civita connection.

The above Proposition 6.4.5 is quite helpful if one wants to calculate $H_{g, \mathcal{S}}$, and the bundle is trivializable. Instead of having to find a harmonic representative of \mathcal{S} , which is a difficult global problem (even in the adiabatic limit), one can calculate the Chern–Simons form on a “good” choice of global frame. Given a frame $p : M \rightarrow P$ (or local frames), the connection can be pulled back to a Lie-algebra valued 1-form on the base. This is how local calculations in geometry are usually performed. We will denote this $\Theta_p = p^*\Theta$ and $\Omega_p = p^*\Omega$. Then,

$$p^*\alpha(\Theta) = \langle \Theta_p \wedge \Omega_p \rangle_{so(\mathfrak{g})} - \frac{1}{6} \langle \Theta_p \wedge [\Theta_p \wedge \Theta_p] \rangle_{so(\mathfrak{g})}.$$

In the following calculation, G will be the base space, and the fiber will be $\text{Spin}(\mathfrak{g}) \cong \text{Spin}(n)$ where $n = \dim G$.

Proposition 6.5.1. *Let G be any compact Lie group with bi-invariant metric g . Let $Spin(G) \rightarrow G$ be a principal $Spin(\mathfrak{g})$ frame bundle for TG with the Levi-Civita connection Θ_g . If $\mathcal{L} \in H^3(Spin(G); \mathbb{Z})$ is the string structure induced by left-invariant framing \mathcal{L} , and $\mathcal{R} \in H^3(Spin(G); \mathbb{Z})$ is the string structure induced by right-invariant framing \mathcal{R} , then*

$$H_{\mathcal{L}} = -\frac{1}{12} \langle ad(\theta) \wedge ad[\theta \wedge \theta] \rangle_{so(\mathfrak{g})}$$

$$H_{\mathcal{R}} = \frac{1}{12} \langle ad(\theta) \wedge ad[\theta \wedge \theta] \rangle_{so(\mathfrak{g})}$$

where $\theta : TG \rightarrow \mathfrak{g}$ is the Maurer-Cartan 1-form.

Proof. Let \mathfrak{g} denote the Lie algebra of left-invariant vector fields on G . Under the left-invariant framing, we have the isomorphism

$$TG \cong G \times T_e G,$$

where $e \in G$ is some point. Any constant map $G \rightarrow T_e G$ in the above isomorphism is a left-invariant vector field, so it is convenient to compose with θ . This gives the trivialization

$$TG \xrightarrow{\approx} G \times T_e G \xrightarrow{G \times \theta} G \times \mathfrak{g}.$$

We now calculate the connection and curvature forms on this frame, denoted $\Theta_{\mathcal{L}}$ and $\Omega_{\mathcal{L}}$. Let X, Y, Z be left-invariant vector fields, and hence constant sections of $G \times T_e G$. Then, the Levi-Civita connection ∇ on TG is given by the formula

$$2\langle \nabla_X Y, Z \rangle = \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle$$

$$= \langle [X, Y], Z \rangle$$

$$\nabla_X Y = \frac{1}{2} [X, Y]$$

We conveniently write the connection as

$$\Theta_{\mathcal{L}} = \frac{1}{2}ad(\theta) \in \Omega^1(G; so(\mathfrak{g})),$$

where $ad : \mathfrak{g} \rightarrow so(\mathfrak{g})$ is the adjoint representation. To calculate the curvature on left-invariant vector fields X, Y, Z ,

$$\begin{aligned} R_{X,Y}Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z \\ &= \frac{1}{4}[X, [Y, Z]] - \frac{1}{4}[Y, [X, Z]] - \frac{1}{2}[[X, Y], Z] \\ &= \frac{1}{4}[[X, Y], Z] - \frac{1}{2}[[X, Y], Z] \\ &= -\frac{1}{4}[[X, Y], Z] \end{aligned}$$

Equivalently, we could use the relation

$$\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta].$$

On vectors $X, Y \in \mathfrak{g}$

$$d\Theta_{\mathcal{L}}(X, Y) = -\frac{1}{2}ad[X, Y],$$

and hence $d\Theta_{\mathcal{L}} = -\frac{1}{4}ad[\theta \wedge \theta]$. Therefore,

$$\Omega_{\mathcal{L}} = -\frac{1}{4}ad[\theta \wedge \theta] + \frac{1}{8}ad[\theta \wedge \theta] = -\frac{1}{8}ad[\theta \wedge \theta].$$

This is equivalent to $\Omega_{\mathcal{L}}(X, Y) = -\frac{1}{4}ad[X, Y]$. Therefore,

$$\begin{aligned} p^* \alpha(\Theta) &= \langle \Theta_{\mathcal{L}} \wedge \Omega_{\mathcal{L}} \rangle_{so(\mathfrak{g})} - \frac{1}{6} \langle \Theta_{\mathcal{L}} \wedge [\Theta_{\mathcal{L}} \wedge \Theta_{\mathcal{L}}] \rangle_{so(\mathfrak{g})} \\ &= \langle \frac{1}{2}ad(\theta) \wedge -\frac{1}{8}ad[\theta \wedge \theta] \rangle_{so(\mathfrak{g})} - \frac{1}{6} \langle \frac{1}{2}ad(\theta) \wedge ad[\frac{1}{2}\theta \wedge \frac{1}{2}\theta] \rangle_{so(\mathfrak{g})} \\ &= -\frac{1}{12} \langle ad(\theta) \wedge ad[\theta \wedge \theta] \rangle_{so(\mathfrak{g})}. \end{aligned}$$

Due to the Ad-invariance of the 3-form, this evaluates as

$$-\frac{1}{12} \langle ad(\theta) \wedge ad[\theta \wedge \theta] \rangle_{so(\mathfrak{g})}(X, Y, Z) = -\frac{1}{2} \langle ad(\theta(X)) \wedge ad[\theta(Y), \theta(Z)] \rangle_{so(\mathfrak{g})}.$$

By the Ad-invariance of the inner product $\langle \cdot, \cdot \rangle_{so(\mathfrak{g})}$ and the fact that $ad : \mathfrak{g} \rightarrow so(\mathfrak{g})$ is a representation, the form is bi-invariant and hence harmonic. Therefore, by Proposition 6.4.5,

$$H_{\mathcal{L}} = -\frac{1}{12} \langle ad(\theta) \wedge ad[\theta \wedge \theta] \rangle_{so(\mathfrak{g})}.$$

Now, we do the same, but with right-invariant framing. Right-multiplication gives an isomorphism

$$TG \cong G \times T_e G,$$

and for right-invariant vector fields X, Y, Z (viewed as elements in $T_e G$ under the above isomorphism) we see that

$$\begin{aligned} \langle \nabla_X Y, Z \rangle &= \frac{1}{2} \langle [X, Y], Z \rangle \\ \nabla_X Y &= \frac{1}{2} [X, Y] \\ R_{X,Y} Z &= -\frac{1}{4} [[X, Y], Z]. \end{aligned}$$

The bracket is the bracket on vector fields. Let $\mathfrak{g}_{\mathcal{R}}$ be the Lie algebra of right-invariant vector fields, and let $\theta_{\mathcal{R}} : TG \rightarrow \mathfrak{g}_{\mathcal{R}}$ be the 1-form that associates to any vector the associated right-invariant vector field. Under the isomorphism $\theta_{\mathcal{R}} : T_e G \cong \mathfrak{g}_{\mathcal{R}}$, we have

$$\begin{aligned} \Theta_{\mathcal{R}} &= \frac{1}{2} ad(\theta_{\mathcal{R}}) \in \Omega^1(G; so(\mathfrak{g}_{\mathcal{R}})), \\ \Omega_{\mathcal{R}} &= -\frac{1}{8} ad[\theta_{\mathcal{R}}, \theta_{\mathcal{R}}] \in \Omega^2(G; so(\mathfrak{g}_{\mathcal{R}})), \end{aligned}$$

where $ad : \mathfrak{g}_{\mathcal{R}} \rightarrow so(\mathfrak{g}_{\mathcal{R}})$ is the adjoint representation. Therefore,

$$\alpha(\Theta)_{\mathcal{R}} = -\frac{1}{12} \langle ad(\theta_{\mathcal{R}}) \wedge [ad(\theta_{\mathcal{R}}) \wedge ad(\theta_{\mathcal{R}})] \rangle_{so(\mathfrak{g}_{\mathcal{R}})} \in \Omega^3(G; so(\mathfrak{g}_{\mathcal{R}})).$$

This form is harmonic, and hence equals $\alpha(\Theta)_{\mathcal{R}}$. To compare this with $\alpha(\Theta)_{\mathcal{L}}$, we remember the following relationship between left and right-invariant vector fields.

If $X \in T_e G$, and $X_L = \theta(X)$, $X_R = \theta_{\mathcal{R}}(X)$ are the associated left and right-invariant vector fields, then

$$[X_L, Y_L]|_e = -[X_R, Y_R]|_e \in T_e G.$$

In other words, the two Lie brackets defined on the tangent space $T_e G$ differ from each other by a sign. Consequently, if we consider $X_e, Y_e, Z_e \in T_e G$, as tangent vectors at a point, $X_L, Y_L, Z_L \in \mathfrak{g}$ the induced left-invariant vector fields, and $X_R, Y_R, Z_R \in \mathfrak{g}_{\mathcal{R}}$ the induced right-invariant vector fields, we see that

$$\begin{aligned} \alpha(\Theta)_{\mathcal{R}}(X_e, Y_e, Z_e) &= -\frac{1}{2} \langle ad(X_R), ad[Y_R, Z_R] \rangle_{so(\mathfrak{g}_{\mathcal{R}})} = -\frac{1}{2} \langle ad(X_L), -ad[Y_L, Z_L] \rangle_{so(\mathfrak{g})} \\ &= \frac{1}{2} \langle ad(X_L), ad[Y_L, Z_L] \rangle_{so(\mathfrak{g})}. \end{aligned}$$

Therefore,

$$\alpha(\Theta)_{\mathcal{R}} = -\alpha(\Theta)_{\mathcal{L}} = \frac{1}{12} \langle ad(\theta) \wedge ad[\theta \wedge \theta] \rangle_{so(\mathfrak{g})} \in \Omega^3(M).$$

□

CHAPTER 7

CANONICAL METRIC CONNECTIONS ASSOCIATED TO STRING STRUCTURES

Suppose we fix a closed compact manifold M along with a choice of spin structure. Assume that M admits a string structure; i.e. that $\frac{p_1}{2}(M) = 0 \in H^4(M; \mathbb{Z})$. Then, given any Riemannian metric g , we can form the $Spin(n)$ frame bundle $Spin(M) = Spin(TM)$. The Levi-Civita connection induces a canonical connection on this bundle, which we will denote Θ_g . By the construction in 6.4.4, we have a canonical map

$$\begin{aligned} Met(M) \times \{\text{string structures on } M\} &\longrightarrow \Omega^3(M) \\ g, \mathcal{S} &\longmapsto H_{g,\mathcal{S}} \end{aligned}$$

The form $H_{g,\mathcal{S}}$ is the canonical 3-form given by the equation (based on $(M, Spin(M), g, \Theta_g)$)

$$\lim_{\delta \rightarrow 0} [\mathcal{S}]_{g\delta} = \alpha(\Theta_g) - \pi^* H_{g,\mathcal{S}}.$$

Remark 7.0.2. *We don't really have the space $Met(M) \times \{\text{string structures on } M\}$. The space of string structures depends on the principal bundle, which varies over the space of metrics. We really mean points in a fiber bundle over $Met(M)$. However, since the fibers are discrete and the space of metrics is contractible, choosing a metric*

g' gives a canonical trivialization of the bundle as

$$\text{Met}(M) \times \{\text{string structures on } \text{Spin}(M, g')\}$$

In general, the space of connections on a principal G -bundle is an affine space modeled on $\Omega^1(M; \mathfrak{g}_P)$, i.e. any two connections differ by a 1-form with values in the adjoint bundle. When the bundle $P = SO(M)$ or $\text{Spin}(M)$,

$$\Omega^1(M; \mathfrak{g}_P) = \Omega^1(M; \text{so}(TM))$$

where elements of $\text{so}(TM)$ are skew-symmetric endomorphisms of the tangent space at each point. Given a 3-form on $H \in \Omega^3(M)$, the metric can be used to create an endomorphism-valued 1-form $T \in \Omega^1(M; \text{so}(TM))$. This follows from the homomorphism

$$\Lambda^3 TM^* \longrightarrow TM^* \otimes \Lambda^2 TM^* \xrightarrow{g} TM^* \otimes \text{so}(TM)$$

induced by the metric g . More explicitly, for any vector field $Z \in C^\infty(M, TM)$,

$$\iota_Z H \in \Omega^2(M) \xrightarrow{g} \text{so}(TM).$$

Therefore, given $H_{g,s} \in \Omega^3(M)$, we define $T = T^{g,s} \in \Omega^1(M; \text{so}(M))$ by

$$\langle T^{g,s}(X, Y), Z \rangle = H_{g,s}.$$

Since H is skew-symmetric in all 3-variables, by definition we see that

$$\langle T(X, Y), Z \rangle = \langle T(Y, Z), X \rangle = -\langle T(Y, X), Z \rangle.$$

In other words, the tensor $g(T(\cdot, \cdot), \cdot)$ is skew-symmetric in all 3-variables, or totally skew-symmetric.

Given $H_{g,s} \in \Omega^3(M)$, we can form the connection

$$\Theta_{g,s} = \Theta_g + \frac{1}{2}T^{g,s}.$$

We will be more concerned with the induced vector bundle connection $\nabla^s = \nabla^{g,s}$ on TM , which we denote

$$\nabla^s = \nabla^g + \frac{1}{2}T^{g,s}.$$

This is the unique metric connection whose torsion T satisfies

$$\langle T(X, Y), Z \rangle = H_{g,s}(X, Y, Z).$$

Proposition 7.0.3. *A metric connection on TM is completely characterized by its torsion.*

Proof. For X, Y, Z vector fields on M , then

- (1) $X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$
- (2) $Y\langle Z, X \rangle = \langle \nabla_Y Z, X \rangle + \langle Z, \nabla_Y X \rangle$
- (3) $Z\langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle$

Taking (1) + (2) - (3) and using $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ we get

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle &= X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle \\ &\quad + \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle \\ &\quad + \langle T(X, Y), Z \rangle - \langle T(Y, Z), X \rangle + \langle T(Z, X), Y \rangle. \end{aligned}$$

Therefore, the connection is completely determined by the metric and torsion tensor.

Furthermore, if we assume that T is totally skew-symmetric, then

$$\begin{aligned}
2\langle \nabla_X Y, Z \rangle &= X\langle Y, Z \rangle + Y\langle X, Z \rangle - Z\langle X, Y \rangle \\
&\quad + \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle \\
&\quad + \langle T(X, Y), Z \rangle \\
\langle \nabla_X Y, Z \rangle &= \langle \nabla_X^g Y + \frac{1}{2}T(X, Y), Z \rangle
\end{aligned}$$

□

In the above construction, the form $H_{g,\mathcal{S}}$ was constructed as a harmonic representative in a scaling limit, and hence the form is *independent* of the global scaling on M . However, we use the metric to transform $H_{g,\mathcal{S}}$ into an endomorphism-valued 1-form. Therefore, the torsion tensor we create *depends* on a global scaling. In fact, if we let

$$g' = \epsilon g, \quad \epsilon \geq 0,$$

then the torsion tensor $T^{g,\mathcal{S}}$ transforms as

$$\epsilon \langle T^{\epsilon g, \mathcal{S}}(X, Y), Z \rangle_g = \langle T^{\epsilon g, \mathcal{S}}(X, Y), Z \rangle_{\epsilon g} = H_{g, \mathcal{S}}(X, Y, Z) = \langle T^{g, \mathcal{S}}(X, Y), Z \rangle_g.$$

Therefore,

$$T^{\epsilon g, \mathcal{S}} = \frac{1}{\epsilon} T^{g, \mathcal{S}}.$$

Consequently, as $\epsilon \rightarrow \infty$, or the manifold becomes scaled very large, the torsion tensor goes to 0. In the opposite scaling, as M becomes very small, the torsion tensor (unless it is zero), becomes very large and does not even converge to an element of $\Omega^1(M; so(TM))$.

Combining the above, for any Riemannian manifold (M, g) with string structure $\mathcal{S} \in H^3(Spin(M); \mathbb{Z})$, we have a *canonical* 1-parameter family of metric connections

given by

$$\Theta_{\epsilon g, \mathcal{S}} = \Theta_{\epsilon g} + \frac{1}{2}T^{\epsilon g, \mathcal{S}},$$

$$\langle \nabla_X^{\epsilon g, \mathcal{S}} Y, Z \rangle_{\epsilon g} = \langle \nabla_X^{\epsilon g} Y, Z \rangle_{\epsilon g} + \frac{1}{2}H_{g, \mathcal{S}}(X, Y, Z).$$

Since the connection ∇^g is invariant under rescaling, it is more convenient to consider this as a 1-paramter family of connections on the fixed metric g . In other words

$$\langle \nabla_X^{\epsilon g, \mathcal{S}} Y, Z \rangle_g = \langle \nabla_X^g Y, Z \rangle_g + \frac{1}{2\epsilon}H_{g, \mathcal{S}}(X, Y, Z).$$

At this point, we will deal with a fixed metric and string structure. The corresponding notation will be dropped when it is unambiguous. In particular, $T = T^{g, \mathcal{S}}$ will be the torsion tensor of the modified connection, and, to avoid excessive parenthesis, we denote

$$T_X Y = T(X, Y).$$

Proposition 7.0.4. *For the connection $\nabla^{\mathcal{S}} = \nabla^g + \frac{1}{2}T$, where $T = T^{g, \mathcal{S}}$,*

$$Ric^{\mathcal{S}}(X, Y) = Ric^g(X, Y) + \frac{1}{2} \sum_i \langle (\nabla_{e_i}^g T)(X, Y), e_i \rangle - \frac{1}{4} \sum_i \langle T_{e_i} X, T_{e_i} Y \rangle.$$

For $\nabla^{\epsilon g, \mathcal{S}} = \nabla^g + \frac{1}{2\epsilon}T$,

$$Ric^{\epsilon g, \mathcal{S}}(X, Y) = Ric^g(X, Y) + \frac{1}{2\epsilon} \sum_i \langle (\nabla_{e_i}^g T)(X, Y), e_i \rangle - \frac{1}{4\epsilon^2} \sum_i \langle T_{e_i} X, T_{e_i} Y \rangle.$$

Proof. Let R denote the curvature tensor. Then,

$$\begin{aligned}
R_{X,Y}^S Z &= \nabla_X^S \nabla_Y^S Z - \nabla_Y^S \nabla_X^S Z - \nabla_{[X,Y]}^S Z \\
&= (\nabla_X^g + \frac{1}{2}T_X)(\nabla_Y^g + \frac{1}{2}T_Y)Z - (\nabla_Y^g + \frac{1}{2}T_Y)(\nabla_X^g + \frac{1}{2}T_X)Z - (\nabla_{[X,Y]}^g + \frac{1}{2}T_{[X,Y]})Z \\
&= \left(\nabla_X^g \nabla_Y^g Z - \nabla_Y^g \nabla_X^g Z - \nabla_{[X,Y]}^g Z \right) + \frac{1}{2} (\nabla_X^g T_Y Z - T_Y \nabla_X^g Z) \\
&\quad - \frac{1}{2} (\nabla_Y^g T_X Z - T_X \nabla_Y^g Z) + \frac{1}{4} (T_X T_Y Z - T_Y T_X Z) - \frac{1}{2} T_{[X,Y]} Z \\
&= R_{X,Y}^g Z + \frac{1}{2} \left((\nabla_X^g T)(Y, Z) + T_{\nabla_X^g Y} Z \right) - \frac{1}{2} \left((\nabla_Y^g T)(X, Z) + T_{\nabla_Y^g X} Z \right) \\
&\quad + \frac{1}{4} (T_X T_Y Z - T_Y T_X Z) - \frac{1}{2} T_{[X,Y]} Z
\end{aligned}$$

Let $\{e_i\}$ be an orthonormal frame. The Ricci tensor, denoted Ric , is defined by

$$Ric(Y, Z) = \sum_{i=1}^n \langle R_{e_i, Y} Z, e_i \rangle.$$

Using the fact that $\langle T_X Y, Z \rangle$ is skew-symmetric in all 3-variables, we see

$$\begin{aligned}
\langle R_{e_i, Y}^S Z, e_i \rangle &= \langle R_{e_i, Y}^g Z, e_i \rangle + \frac{1}{2} \langle (\nabla_{e_i}^g T)(Y, Z), e_i \rangle + \frac{1}{2} \langle T_{\nabla_{e_i}^g Y} Z, e_i \rangle - \frac{1}{2} \langle (\nabla_Y^g T)(e_i, Z), e_i \rangle \\
&\quad - \frac{1}{2} \langle T_{\nabla_Y^g e_i} Z, e_i \rangle + \frac{1}{4} \langle T_{e_i} T_Y Z, e_i \rangle - \frac{1}{4} \langle T_Y T_{e_i} Z, e_i \rangle - \frac{1}{2} \langle T_{[e_i, Y]} Z, e_i \rangle \\
&= \langle R_{e_i, Y}^g Z, e_i \rangle + \frac{1}{2} \langle (\nabla_{e_i}^g T)(Y, Z), e_i \rangle + \frac{1}{2} \langle T_{\nabla_{e_i}^g Y} Z - T_{\nabla_Y^g e_i} Z - T_{[e_i, Y]} Z, e_i \rangle \\
&\quad - \frac{1}{2} \langle (\nabla_Y^g T)(e_i, Z), e_i \rangle - \frac{1}{4} \langle T_Y T_{e_i} Z, e_i \rangle \\
&= \langle R_{e_i, Y}^g Z, e_i \rangle + \frac{1}{2} \langle (\nabla_{e_i}^g T)(Y, Z), e_i \rangle - \frac{1}{4} \langle T_{e_i} Y, T_{e_i} Z \rangle
\end{aligned}$$

The term $\frac{1}{4} \langle T_{e_i} T_Y Z, e_i \rangle$ drops out in the second equality due to the skew-symmetry of $\langle T(\cdot, \cdot), \cdot \rangle$. In the last equality, we have that

$$\frac{1}{2} \langle T_{\nabla_{e_i}^g Y} Z - T_{\nabla_Y^g e_i} Z - T_{[e_i, Y]} Z, e_i \rangle = \frac{1}{2} \langle T_{T_{e_i} Y} Z, e_i \rangle = 0$$

due to the torsion-freeness of the Levi-Civita connection. Also,

$$\begin{aligned}
\langle (\nabla_{e_i}^g T)(Y, Z), e_i \rangle &= \langle \nabla_Y^g T_{e_i} Z, e_i \rangle - \langle T_{\nabla_Y^g e_i} Z, e_i \rangle - \langle T_{e_i} \nabla_Y^g Z, e_i \rangle \\
&= \langle \nabla_Y^g T_{e_i} Z, e_i \rangle + \langle T_{e_i} Z, \nabla_Y^g e_i \rangle \\
&= Y \langle T_{e_i} Z, e_i \rangle = 0
\end{aligned}$$

Therefore, we have the relationship

$$Ric^{\mathcal{S}}(X, Y) = Ric^g(X, Y) + \frac{1}{2} \sum_i \langle (\nabla_{e_i}^g T)(X, Y), e_i \rangle - \frac{1}{4} \sum_i \langle T_{e_i} X, T_{e_i} Y \rangle.$$

If one keeps track of the ϵ in the above calculation for $\nabla^{\epsilon g, \mathcal{S}} = \nabla^g + \frac{1}{2\epsilon} T$, then

$$Ric^{\epsilon g, \mathcal{S}}(X, Y) = Ric^g(X, Y) + \frac{1}{2\epsilon} \sum_i \langle (\nabla_{e_i}^g T)(X, Y), e_i \rangle - \frac{1}{4\epsilon^2} \sum_i \langle T_{e_i} X, T_{e_i} Y \rangle.$$

□

The modified Ricci tensor contains the original Ricci tensor, plus two other terms. The first is skew-symmetric, and the second is symmetric. In particular, if we look at $Ric(X) = Ric(X, X)$, then

$$Ric^{\epsilon g, \mathcal{S}}(X) = Ric^g(X) - \frac{1}{4\epsilon^2} \sum_i \|T_{e_i} X\|^2. \quad (7.0.1)$$

Consequently, if $H_{\mathcal{S}} \neq 0$, then for some X ,

$$Ric^{\epsilon g, \mathcal{S}}(X) \xrightarrow{\epsilon \rightarrow 0} -\infty.$$

Conversely, if the 1-parameter family of modified connections have positive Ricci curvature for ϵ close to 0 (i.e. if $Ric^{\epsilon g, \mathcal{S}} > 0$ for $\epsilon \sim 0$), then $Ric^{\epsilon, \mathcal{S}} > 0$ for all ϵ and $H_{\mathcal{S}} = 0$. In this situation, the 1-parameter family of metrics is really the Levi-Civita connection for a metric of positive Ricci curvature.

Remark 7.0.5. *It was originally hoped that given (M, g, \mathcal{S}) , the canonical metric connection $\nabla^{g, \mathcal{S}}$ would have its modified Pontryagin form $\frac{\rho_1}{2}(\nabla^{g, \mathcal{S}}) = 0$. The above*

discussion shows this can not be true, since the Pontryagin form from the Levi-Civita connection is invariant under a global rescaling, and the connection $\nabla^{\epsilon g, \mathbb{S}}$ converges to the Levi-Civita connection as $\epsilon \rightarrow \infty$. Therefore, if $\frac{p_1}{2}(M, g) \neq 0 \in \Omega^4(M)$, the modified Pontryagin form can not always be zero. One might hope that in the other scaling limit, as $\epsilon \rightarrow 0$, the modified Pontryagin form is zero. However, one can show this is not the case either. This can be seen by a brute force construction on $S^1 \times SU(2)$, where the metric is defined as a 1-parameter family of left-invariant metrics on $SU(2)$.

BIBLIOGRAPHY

- [AHS] M. Ando, M. J. Hopkins, and N. P. Strickland. Elliptic spectra, the Witten genus and the theorem of the cube. *Invent. Math.*, 146(3):595–687, 2001.
- [AS1] Orlando Alvarez and I. M. Singer. Beyond the elliptic genus. *Nuclear Phys. B*, 633(3):309–344, 2002.
- [AS2] M. F. Atiyah and L. Smith. Compact Lie groups and the stable homotopy of spheres. *Topology*, 13:135–142, 1974.
- [BF] Jean-Michel Bismut and Daniel S. Freed. The analysis of elliptic families. II. Dirac operators, eta invariants, and the holonomy theorem. *Comm. Math. Phys.*, 107(1):103–163, 1986.
- [CE] Claude Chevalley and Samuel Eilenberg. Cohomology theory of Lie groups and Lie algebras. *Trans. Amer. Math. Soc.*, 63:85–124, 1948.
- [CP] R. Coquereaux and K. Pilch. String structures on loop bundles. *Comm. Math. Phys.*, 120(3):353–378, 1989.
- [CS] Shiing Shen Chern and James Simons. Characteristic forms and geometric invariants. *Ann. of Math. (2)*, 99:48–69, 1974.
- [Dai] Xianzhe Dai. Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence. *J. Amer. Math. Soc.*, 4(2):265–321, 1991.
- [For] Robin Forman. Spectral sequences and adiabatic limits. *Comm. Math. Phys.*, 168(1):57–116, 1995.
- [Fre1] Daniel S. Freed. Classical Chern-Simons theory. I. *Adv. Math.*, 113(2):237–303, 1995.
- [Fre2] Daniel S. Freed. Classical Chern-Simons theory. II. *Houston J. Math.*, 28(2):293–310, 2002.
- [Hit] Nigel Hitchin. Harmonic spinors. *Advances in Math.*, 14:1–55, 1974.
- [Hop] M. J. Hopkins. Algebraic topology and modular forms. In *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)*, pages 291–317, Beijing, 2002. Higher Ed. Press.
- [KN] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol I*. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963.

- [Lic] André Lichnerowicz. Laplacien sur une variété riemannienne et spineurs. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)*, 33:187–191, 1962.
- [LM] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.
- [MM] Rafe R. Mazzeo and Richard B. Melrose. The adiabatic limit, Hodge cohomology and Leray’s spectral sequence for a fibration. *J. Differential Geom.*, 31(1):185–213, 1990.
- [Seg1] Graeme Segal. Elliptic cohomology (after Landweber-Stong, Ochanine, Witten, and others). *Astérisque*, (161-162):Exp. No. 695, 4, 187–201 (1989), 1988.
- [Seg2] Graeme Segal. The definition of conformal field theory. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 421–577. Cambridge Univ. Press, Cambridge, 2004.
- [ST] Stephan Stolz and Peter Teichner. What is an elliptic object? In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 247–343. Cambridge Univ. Press, Cambridge, 2004.
- [Sto] Stephan Stolz. A conjecture concerning positive Ricci curvature and the Witten genus. *Math. Ann.*, 304(4):785–800, 1996.
- [Wit1] Edward Witten. Global gravitational anomalies. *Comm. Math. Phys.*, 100(2):197–229, 1985.
- [Wit2] Edward Witten. The index of the Dirac operator in loop space. In *Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986)*, volume 1326 of *Lecture Notes in Math.*, pages 161–181. Springer, Berlin, 1988.
- [Wit3] Edward Witten. Index of Dirac operators. In *Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997)*, pages 475–511. Amer. Math. Soc., Providence, RI, 1999.